Exercise sheet 2 - Quantum mechanics

1. The state space of a certain physical system is three-dimensional. Let $\{|u_1\rangle, |u_2\rangle, |u_3\rangle\}$ be an orthonormal basis of this space. The kets $|\psi_0\rangle$ and $|\psi_1\rangle$ are defined by

$$
\begin{aligned}
|\psi_0\rangle &= \frac{1}{\sqrt{2}}|u_1\rangle + \frac{i}{2}|u_2\rangle + \frac{1}{2}|u_3\rangle \\
|\psi_1\rangle &= \frac{1}{\sqrt{3}}|u_1\rangle + \frac{i}{\sqrt{3}}|u_3\rangle
\end{aligned}
$$

   (a) Are these kets normalized?Justify your answer.

2. Let $\hat{K}$ be an operator defined by $\hat{K} = |\varphi\rangle\langle\psi|$, where $|\varphi\rangle$ and $|\psi\rangle$ are two vectors of the state space.

   (a) Under which conditions is $\hat{K}$ Hermitian?

   (b) Compute $\hat{K}^2$. Under which condition is $\hat{K}$ a projector?

3. Let $|\phi_n\rangle$ $(n = 1, 2, ...)$ be a complete set of orthonormal states, and let $\hat{A}$ be an operator. The matrix elements of the operator $\hat{A}$ are defined to be

$$(A)_{ij} = \langle\phi_i|\hat{A}|\phi_j\rangle.$$

   The matrix element of the Hermitian conjugate $\hat{A}^\dagger$ are defined to be

$$(A^\dagger)_{ij} = \langle\phi_i|\hat{A}^\dagger|\phi_j\rangle.$$

   Show that the relationship between the matrix elements $(A)_{ij}$ and $(A^\dagger)_{ij}$ is

$$(A^\dagger)_{ij} = (A)^*_{ji}.$$

   Hence, show that for a Hermitian operator

$$(A)_{ij} = (A)^*_{ji}.$$

4. Let $\hat{A}$ and $\hat{B}$ be two Hermitian operators. Show that:

   (a) The operators $\hat{A}\hat{B} + \hat{B}\hat{A}$ and $i(\hat{A}\hat{B} - \hat{B}\hat{A})$ are Hermitian

   (b) The expectation value of $\hat{A}^2$ in any state is always real and non-negative

   (c) If $\hat{A}^2 = \hat{A}$, the only possible eigenvalues of $\hat{A}$ are 0 and 1.