Answer THREE questions.

The numbers in square brackets in the right-hand margin indicate the provisional allocation of maximum marks per sub-section of a question.

1. The WKB approximation for the wavefunction when $E > V(x)$ is $\Psi(x) \simeq \frac{1}{\sqrt{k(x)}} \exp\left[\pm i \int^x k(x')dx'\right]$. Discuss the regime of validity of the WKB approximation and explain why the expression above would not be valid close to a classical turning point. Give the corresponding form when $E < V(x)$. [4]

   Outline a method whereby the two types of solution might be linked. Do not give extensive mathematical details. Do not derive connection formulae. [4]

   Quantum particles of energy E approach (from $x = -\infty$) a barrier where the potential, $V(x)$, is of the form:
   if $-a > x$ $V(x) = 0$,
   if $|a| \geq x$ $V(x) = V_0(1 - |x|/a)$ and
   if $a < x$ $V(x) = 0$.
   where $V_0 > 0$ is a constant.

   Obtain the turning points $x = t_1$ and $x = t_2$, where $t_2 > 0$, as a function of E. [2]

   Explain why the WKB wavefunction, in the classically allowed region, $(E > V(x))$, to the *right* of the barrier takes the form: $\Psi_3 \simeq \frac{A}{\sqrt{k(x)}} \exp\left[i \int_{t_2}^{x} k(x')dx'\right]$ and outline briefly how the corresponding form within the barrier,
   $\Psi_2 \simeq \frac{A}{2\sqrt{q(x)}} \exp\left[- \int_x^{t_2} q(x')dx'\right] - \frac{iA}{\sqrt{q(x)}} \exp\left[\int_x^{t_2} q(x')dx'\right]$ might be obtained. . [3]

   Far to the left of the barrier, we can show that this connects with an incident wave of the form : $\Psi_{inc} \simeq \frac{-Ai}{\sqrt{k(x)}}(\frac{1}{4r} + r) \exp\left[-i \int_x^{t_1} k(x')dx'\right]$, where
   $r = \exp\left[\, {}^{t_2}_{t_1} \int q(x')dx'\right] = e^{\lambda}$.
   This corresponds asymptotically to free particles incident from the left. Show that, in the WKB regime (r large) the tunnelling probability
   $T = \frac{1}{(\frac{1}{4r}+r)^2} \simeq e^{-2\lambda}$. [3]

   For the particular barrier given above, calculate the tunnelling probability as a function of energy. You may use the integral $\int (1-y)^{1/2}\, dy = -\frac{2}{3}(1-y)^{3/2}$. [4]

2. The time-evolution of a quantum system is given by the time-dependent Schrödinger equation, $i\hbar\frac{\partial\psi}{\partial t} = H\psi$. An alternative formulation employs a time evolution operator $T(t, t_0)$ which satisfies a differential equation of the form: $i\hbar\frac{\partial T(t,t_0)}{\partial t} = HT(t, t_0)$. Show that the form appropriate for an infinitesimal time interval δt is:

$$T(t + \delta t, t) = \exp -\frac{i}{\hbar}H(t)\delta t.$$

Obtain also the form of $T(t, t_0)$ appropriate for a time-independent Hamiltonian. [5]

A quantum particle evolves in a time-periodic potential $V(x, t) = V(x, t+\tau)$, with period τ, too strong to be treated perturbatively. Its Floquet states take the form $\Psi_n(x, t) = \exp -i\epsilon_n t\, U_n(x, t)$ where $U_n(x, t) = U_n(x, t+\tau)$ and ϵ_n is a quasi-energy.

Explain how Floquet states may be used to evolve a general quantum state in a time-periodic potential. You should explain how they relate to $T(t, t_0)$. [5]

The Hamiltonian of the particle $H = H_0+V(x, t)$, is the sum of a time-independent part, H_0, and a time-periodic potential $V(x, t)$.

Show, from the time-dependent Schrödinger equation, that we can calculate $U_n(x, t)$ from the eigenvalue equation:

$$FU_n(x, t) = \hbar\epsilon_n U_n(x, t)$$

Where the Floquet operator $F = H - i\hbar\frac{\partial}{\partial t}$. [4]

The particle has a time-independent Hamiltonian H_0 with eigenfunctions $\psi_n(x) = \frac{1}{\sqrt{2\pi}}e^{inx}$ where $n = 0, \pm1, \pm2...$, the coordinate $0 \leq x \leq 2\pi$ and $H_0\psi_n(x) = \psi_n(x)E_n$. The particle is also subjected to a strong laser field of form $V(x, t) = A \sin x \cos \Omega t$, corresponding to period $\tau = 2\pi/\Omega$.

We expand our Floquet states in a complete basis of orthonormal states, so $U_n(x, t) = \sum_{j,m} C^n_{j,m}\psi_j(x) \exp im\Omega t$. Using this basis, calculate the form of the matrix elements, $< jm|F|j'm' >$, of the Floquet operator. [6]

3. A system subjected to a time-dependent perturbation is described by a Hamiltonian:

$$H(\mathbf{r}, t) = H_0(\mathbf{r}) + \lambda V'(\mathbf{r}, t)$$

where λ is a small parameter. The eigenfunctions $\psi_n^{(0)}(\mathbf{r})$ and eigenvalues E_n of $H_0(\mathbf{r})$ are known. Given that a solution of the time-dependent Schrödinger equation can be written as:

$$\Psi(\mathbf{r}, t) = \sum_n c_n(t)\psi_n^{(0)}(\mathbf{r}) \exp(-iE_n t/\hbar)$$

obtain a differential equation for the transition coefficients $c_n(t)$. [4]

Initially, at time t_0, the system is in a definite eigenstate $\psi_i^{(0)}(\mathbf{r})$. Show that at a later time t, to lowest order in λ, the transition amplitude for excitation of a state $\psi_k(\mathbf{r})$ of energy E_k ($\neq E_i$) is given by

$$c_k(t) = \frac{1}{i\hbar}\int_{t_0}^{t} < \psi_k^{(0)}(\mathbf{r})|\lambda V'(\mathbf{r}, t)|\psi_i^{(0)}(\mathbf{r}) > e^{i\omega_{ki}t'}dt'$$

where $\omega_{ki} = (E_k - E_i)/\hbar$. [5]

Show that the case $i = k$ corresponds to a simple phase shift on the initial eigenfunction. [2]

A particle in a magnetic field has eigenfunctions $\psi_n(\phi) = \frac{1}{\sqrt{2\pi}}e^{in\phi}$ where $n = 0, \pm 1, \pm 2...$, the coordinate $0 \leq \phi \leq 2\pi$ and the energies $E_n = n\epsilon$. At $t \leq 0$ the particle is in the eigenstate corresponding to $n = 2$. At $t > 0$ it is acted on by a weak perturbation

$$\lambda V(\mathbf{r}, t) = B \sin\phi \exp -\gamma t$$

while for time $t > T$, the perturbation is turned off and $\lambda V(\mathbf{r}, t) = 0$. B and γ are constants.

Calculate which transitions are allowed and which are forbidden. [3]

Calculate the probabilities, for all allowed transitions, that at later time $t > T$, the particle is in a state $n \neq 2$ [3]

Transitions to $n = 0$ are forbidden to first order. Discuss briefly how this might, however, be possible to second order. [3]

4. Explain the significance of a wavefunction of the form:

$$\psi(\mathbf{r}) = \frac{1}{2k}\sum_{l=0}^{\infty}(2l+1)\ i^{l+1}\left[\frac{e^{-i(kr-l\pi/2)}}{r} - S_l\ \frac{e^{+i(kr-l\pi/2)}}{r}\right] P_l(\cos\theta)$$

in studies of quantum scattering. You should explain all the terms on the right hand side and discuss their physical significance, stating the regime of validity of such a wavefunction. Suggest a form of S_l for the case of elastic scattering. Suggest also a modification appropriate for inelastic scattering. [5]

Modify the right-hand side of the above expression for the case where there is no interaction potential, hence where the wavefunction $\psi(\mathbf{r})$ corresponds to a simple plane wave e^{ikz}. [2]

Hence show that the scattered part of the wavefunction $f(\theta)\ \frac{e^{+ikr}}{r}$ has amplitude:

$$f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l(k)}\sin\delta_l(k)P_l(\cos\theta),$$

where δ_l is the $l-th$ partial-wave phase-shift. [4]

Now show that the total elastic cross-section, σ, is given by:

$$\sigma = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\frac{1}{1+\cot^2\delta_l(k)}.$$

[3]

When might you expect a resonant cross-section? [1]

Obtain an approximate analytical form for the elastic cross section appropriate for low energies, explaining carefully your reasoning. [2]

At low energies the phase-shifts from a scattering experiment are given as $\tan\delta_0(E) = \sqrt{(E/\epsilon)}$. Show that $\sigma \simeq \frac{4\pi\hbar^2}{2m}\frac{1}{E+\epsilon}$. [3]

NOTE : **In answering this question, you may use this result :**
$\int_0^\pi P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta d\theta = \frac{2}{2l+1}\delta_{ll'}$.

5. The Cartesian components of the spin angular momentum operators **S**, of a spin-1/2 particle, satisfy commutation relations $[S_i, S_j] = i\hbar S_k$ and $[S^2, S_i] = 0$, respectively, where i, j, k are cyclic permutations of x, y, z. The raising and lowering operators S_- and S_+ are defined by:

$$S_\pm = S_x \pm iS_y$$

Show that:

$$[S_z, S_\pm] = \pm\hbar S_\pm$$

and,

$$S_\pm S_\mp = S^2 - S_z^2 \pm \hbar S_z$$

[4]

$|sm>= |\frac{1}{2}\ \frac{1}{2}>= |\alpha>$ and $|sm>= |\frac{1}{2}\ -\frac{1}{2}>= |\beta>$ are simultaneous eigenstates of S^2 and S_z such that:

$$S^2|sm>= s(s+1)\hbar^2|sm>$$

and,

$$S_z|sm>= m\hbar|sm>$$

By considering the result $S_\pm|sm>= C_\pm|sm\pm 1>$ and assuming $<\psi|S_\pm|\phi>=<\phi|S_\mp|\psi>^*$, show that $C_\pm = \sqrt{[s(s+1) - m(m\pm 1)]}\hbar$ [4]

S_n is the component of the spin angular momentum operator along the direction of a unit vector $\hat{\mathbf{n}} = (\sin\theta, 0, \cos\theta)$. Show that:

$$S_n = \frac{1}{2}[\sin\theta S_+ + \sin\theta S_- + 2S_z\cos\theta]$$

[3]

Express S_n in matrix form in the basis of the eigenstates $|\alpha>$ and $|\beta>$. [4]

A spin-1/2 particle is subjected to a perturbation characterised by the operator $V = B(S^2 - S_n^2)$ where B is a constant. Work out the expectation value of V for a quantum particle in the state $|\chi>= \frac{1}{\sqrt{2}}(\ |\alpha> + |\beta>)$. [5]

END OF PAPER