Answer THREE questions.

The numbers in square brackets in the right-hand margin indicate the provisional allocation of maximum marks per sub-section of a question.

1. Explain briefly the WKB method and its applications in quantum theory. You should discuss its regime of validity. Explain in your answer the significance of classical turning points and of connection formulae. [4]

   Starting from a wavefunction of the form $\psi(x) = Ae^{iS(x)/\hbar}$, show from the Schrödinger equation that the phase $S(x)$ obeys a differential equation of the form:

$$-\left(\frac{S'}{\hbar}\right)^2 + \frac{iS''}{\hbar} + \frac{P^2(x)}{\hbar^2} = 0.$$

   Give the function $P(x)$ in terms of the energy E and potential energy $V(x)$. [2]

   Hence and by assuming that $\hbar$ is a small quantity, derive the general form of the WKB wavefunction as a sum of terms of the form $\frac{A_\pm}{\sqrt{K(x)}} e^{\pm i \int K(x')dx'}$. [6]

   Give the corresponding form for a classically forbidden region, where $E < V(x)$. [2]

   A quantum particle is in a well with a single rigid wall at $x = 0$.

$$\begin{aligned} x < 0 \quad & V(x) = \infty \\ a \geq x \geq 0 \quad & V(x) = V_0 \; x/a \\ x > a \quad & V(x) = V_0 \end{aligned}$$

   where $V_0 > 0$ is a constant.
   Show that if $\sqrt{2mV_0} = 1$ and $\frac{2a}{3\hbar\pi} = 27$, the eigenenergies E_n of the particle, within the WKB approximation, are given by the relation:

$$\frac{E_n}{V_0} = \frac{1}{9}\left[n + \frac{3}{4}\right]^{2/3}.$$

   [5]

   How many bound states can the well support? [1]

2. Explain briefly TWO of the following three topics:

a) The phenomenon of wavepacket revivals in the context of the time evolution of a system with discrete eigenstates. [4]

b) The Heisenberg Uncertainty Principle, in relation to a Gaussian wavepacket, for which the momentum probability amplitude at $t = 0$ takes the form: $\Phi(p) = A\exp(-\frac{p^2\sigma^2}{\hbar^2})$. Explain (without explicitly deriving it) why this implies $\Psi(x) = B\exp(-\frac{x^2}{4\sigma^2})$. A and B are normalising constants. [4]

c) The role of Floquet theory in studying the time evolution of quantum systems. [4]

The wavefunction $\Psi(t)$ of a quantum system evolves in time according to

$$\Psi(t) = T(t, t_0)\Psi(t_0),$$

where $T(t, t_0)$ is a unitary operator and t_0 is an arbitrary initial time. Starting from the time-dependent Schrödinger equation, show that this time-evolution operator satisfies the differential equation

$$i\hbar\frac{\delta T(t, t_0)}{\delta t} = HT(t, t_0)$$

[3]

Hence obtain the form appropriate for an infinitesimal time interval

$$T(t + \delta t, t) = \exp -\frac{i}{\hbar}H(t)\delta t.$$

[3]

By considering the series expansion, to appropriate order, of an operator of the form $\hat{T} = \exp(\hat{A} + \hat{B})$ where $\hat{A}$ and $\hat{B}$ are non-commuting operators, show that, in the split-operator method, the time-evolution of a wavefunction $\Psi(x, t)$ over a short time interval Δt in a time-dependent potential $V(x, t)$, may be approximated by:

$$\Psi(x, t+\Delta t) \simeq \exp\left(\frac{-i\hat{P}^2}{2m\hbar}\frac{\Delta t}{2}\right) . \exp\left(\frac{-iV(x, t)\Delta t}{\hbar}\right) . \exp\left(\frac{-i\hat{P}^2}{2m\hbar}\frac{\Delta t}{2}\right)\Psi(x, t) + O((\Delta t)^3)$$

[3]

Why is a sequence of Fourier transforms usually employed to implement this method in practice? Give a sequence appropriate for the time-evolution operator corresponding to the example above, using the notation $\psi(x) = FT[\phi(p)]$. [3]

3. A system subjected to a time-dependent perturbation is described by a Hamiltonian:

$$H(\mathbf{r},t) = H_0(\mathbf{r}) + \lambda V'(\mathbf{r},t)$$

where λ is a small parameter. The eigenfunctions $\psi_n^{(0)}(\mathbf{r})$ and eigenvalues E_n of $H_0(\mathbf{r})$ are known. Given that a solution of the time-dependent Schrödinger equation can be written as:

$$\Psi(\mathbf{r},t) = \sum_n c_n(t)\psi_n^{(0)}(\mathbf{r})\exp(-iE_nt/\hbar)$$

obtain a differential equation for the transition coefficients $c_n(t)$. [4]

Initially, at time t_0, the system is in a definite eigenstate $\psi_i^{(0)}(\mathbf{r})$. Show that at a later time t, to lowest order in λ, the transition amplitude for excitation of a state $\psi_k(\mathbf{r})$ of energy E_k ($\neq E_i$) is given by

$$c_k(t) = \frac{1}{i\hbar}\int_{t_0}^{t} < \psi_k^{(0)}(\mathbf{r})|\lambda V'(\mathbf{r},t)|\psi_i^{(0)}(\mathbf{r}) > e^{i\omega_{ki}t'}dt'$$

where $\omega_{ki} = (E_k - E_i)/\hbar$. [5]

Show that the case $i = k$ corresponds to a simple phase shift on the initial eigenfunction. [2]

A particle in a magnetic field has eigenfunctions $\psi_n(\phi) = \frac{1}{\sqrt{2\pi}}e^{in\phi}$ where $n = 0, \pm1, \pm2...$, the coordinate $0 \leq \phi \leq 2\pi$ and the energies $E_n = n\epsilon$. At $t \leq 0$ the particle is in the eigenstate corresponding to $n = 1$. At $t > 0$ it is acted on by a weak perturbation

$$\lambda V'(\mathbf{r},t) = B\cos 2\phi \cos \Omega t$$

Into which two states can transitions be induced, to lowest order in B? [5]

Calculate the transition probability for each of these allowed transitions, taking care to distinguish emission and absorption processes. [4]

4. Explain the significance of the wavefunction:

$$\psi(\mathbf{r}) = \mathrm{e}^{ikz} + f(\theta)\frac{e^{ikr}}{r}$$

in studies of elastic quantum scattering of spinless particles by a central potential of finite range. You should explain all the terms on the right hand side and discuss their physical significance, stating the regime of validity of such a wavefunction. [3]

How would you calculate the differential cross section and the total cross section from the above solution? [2]

Calculate the corresponding flux of incident particles and scattered particles. You may use $\mathbf{J} = Re\{\psi^* \frac{\hbar}{mi}\nabla\psi\}$ and $\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$. [4]

For elastic scattering, the partial wave expansion for $f(\theta)$ is

$$f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l(k)}\sin\delta_l(k)P_l(\cos\theta),$$

where P_l is a Legendre polynomial and δ_l is a partial-wave phase-shift. Show that the total cross-section, σ, is given by:

$$\sigma = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l(k).$$

[3]

Explain the physical significance of the centrifugal barrier and why the general result for elastic scattering $\delta_l(k) \propto k^{2l+1}$ is physically plausible. [3]
Hence derive the approximate form of the cross section at low energy $\sigma = 4\pi A^2$ where A is a scattering length. [2]
For a beam of particles scattered by a repulsive potential α/r^2, it can be shown that

$$\delta_l(k) = -\frac{\pi}{4}\left(\left[(2l+1)^2 + \frac{8m\alpha}{\hbar^2}\right]^{1/2} - (2l+1)\right).$$

Hence show that, if $8m\alpha/\hbar^2 << 1$, then the differential cross section

$$\frac{d\sigma}{d\Omega} \simeq \frac{\pi^2 m^2\alpha^2}{4\hbar^4 k^2}\frac{1}{\sin^2(\theta/2)}.$$

[3]

NOTE : **In answering this question, you may use these results :**
$\sum_{l=0}^{\infty} P_l(\cos\theta) = \frac{1}{2\sin(\theta/2)}$ and $\int_0^{\pi} P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta d\theta = \frac{2}{2l+1}\delta_{ll'}$.

5. The Cartesian components of the orbital and spin angular momentum operators, $\mathbf{L}$ and $\mathbf{S}$ satisfy commutation relations $[L_i, L_j] = i\hbar L_k$ and $[S_i, S_j] = i\hbar S_k$, respectively, where i, j, k are cyclic permutations of x, y, z. In addition $[L_i, \mathbf{L}^2] = [S_i, \mathbf{S}^2] = 0$. The orbital angular momentum and spin couple to give a total angular momentum $\mathbf{L} + \mathbf{S} = \mathbf{J}$.

Show that the operator $\mathbf{L}.\mathbf{S}$ does not commute with L_z or S_z but does commute with $\mathbf{L}^2$, $\mathbf{S}^2$, $\mathbf{J}^2$ and J_z. [5]

Show that $\mathbf{L}.\mathbf{S}$ can be expressed, using quantum raising and lowering operators:

$$\mathbf{L}.\mathbf{S} = \frac{1}{2}[L_+S_- + L_-S_+] + L_zS_z$$

[3]

You may use the tables of Clebsch-Gordan coefficients provided to answer this part of the question.
A quantum particle with $L = 2$ and $S = 1$ is in a total angular momentum state $|LSJM >= |2130 >$.

Calculate the expectation values of the following operators :

i) $< \mathbf{L}.\mathbf{S} >$ [3]
ii) $< L_z >$ [3]
iii) $< L_+S_- + L_-S_+ >$ [3]
iv) $< S_x >$ [3]

Note: For this question, you may use the following relations for a general angular momentum operator $\mathbf{j}$ (valid for all operators $\mathbf{S}, \mathbf{L}, \mathbf{J}$):
$j_\pm = j_x \pm ij_y$ and
$j_\pm|jm >= \sqrt{j(j+1) - m(m \pm 1)}\ \hbar|jm \pm 1 >$.

SEE FORMULA SHEET ATTACHED

END OF PAPER