5.1. In lectures it was stated that the covariant derivative of the metric was zero, $\nabla \mathbf{g} = 0$.

(a) Use this to show the often-useful property that index-raising and lowering operations can be moved through covariant differentiation, for example

$$V_{\alpha;\beta} = (g_{\alpha\gamma}V^{\gamma})_{;\beta} = g_{\alpha\gamma}V^{\gamma}{}_{;\beta}$$

Applying the product rule:

$$V_{\alpha;\beta} = (g_{\alpha\gamma}V^{\gamma})_{;\beta} = g_{\alpha\gamma;\beta}V^{\gamma} + g_{\alpha\gamma}V^{\gamma}{}_{;\beta}.$$

since $\nabla g = 0$, *then* $g_{\alpha\beta;\gamma} = 0$ *and so*

$$V_{\alpha;\beta} = g_{\alpha\gamma}V^{\gamma}{}_{;\beta}$$

QED.

(b) Hence, given the relation

$$V_{\alpha;\beta\gamma} - V_{\alpha;\gamma\beta} = R^{\rho}{}_{\alpha\beta\gamma}V_{\rho}$$

show that

$$V^{\alpha}{}_{;\beta\gamma} - V^{\alpha}{}_{;\gamma\beta} = R_{\rho}{}^{\alpha}{}_{\beta\gamma}V^{\rho}$$

a relation used in lectures when discussing the Riemann tensor.

Changing α *to* σ *and then multiplying by* $g^{\alpha\sigma}$

$$g^{\alpha\sigma}V_{\sigma;\beta\gamma} - g^{\alpha\sigma}V_{\sigma;\gamma\beta} = g^{\alpha\sigma}R^{\rho}{}_{\sigma\beta\gamma}V_{\rho}$$

which from the above result becomes

$$V^{\alpha}{}_{;\beta\gamma} - V^{\alpha}{}_{;\gamma\beta} = R^{\rho\alpha}{}_{\beta\gamma}V_{\rho}$$

Setting $V_{\rho} = g_{\rho\sigma}V^{\sigma}$ *then*

$$V^{\alpha}{}_{;\beta\gamma} - V^{\alpha}{}_{;\gamma\beta} = g_{\rho\sigma}R^{\rho\alpha}{}_{\beta\gamma}V^{\sigma} = R_{\sigma}{}^{\alpha}{}_{\beta\gamma}V^{\sigma}$$

5.2. Calculation of the Riemann tensor in one of the most tedious in GR, however, it is not difficult – in principle – and is worth doing for the simplest case of all, the 2-sphere of radius a, labelled in terms of the spherical polar angles θ and ϕ for which

$$ds^2 = a^2\, d\theta^2 + a^2 \sin^2\theta\, d\phi^2,$$

and the only non-zero connection coefficients are $\Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta$ and $\Gamma^{\phi}{}_{\phi\theta} = \Gamma^{\phi}{}_{\theta\phi} = \cot\theta$ (see problem sheet 4).

Evaluate the Riemann and Ricci tensors, and thus show that the Ricci scalar $R = -2/a^2$.

The (non-zero) metric coefficients can be read from the interval: $g_{\theta\theta} = a^2$, $g_{\phi\phi} = a^2\sin^2\theta$, *while* $g^{\theta\theta} = a^{-2}$ *and* $g^{\phi\phi} = a^{-2}\sin^{-2}\theta$. *The covariant Riemann tensor is given by*

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\rho}\left(\Gamma^{\rho}{}_{\beta\delta,\gamma} - \Gamma^{\rho}{}_{\beta\gamma,\delta} + \Gamma^{\sigma}{}_{\beta\delta}\Gamma^{\rho}{}_{\sigma\gamma} - \Gamma^{\sigma}{}_{\beta\gamma}\Gamma^{\rho}{}_{\sigma\delta}\right).$$

Handout 4 shows that $R_{\alpha\beta\gamma\delta}$ *is anti-symmetric in* α *and* β *and in* γ *and* δ *leaving only components related to* $R_{\theta\phi\theta\phi}$. *Thus*

$$\begin{aligned} R_{\theta\phi\theta\phi} &= g_{\theta\theta}\left(\Gamma^{\theta}{}_{\phi\phi,\theta} - \Gamma^{\theta}{}_{\phi\theta,\phi} + \Gamma^{\sigma}{}_{\phi\phi}\Gamma^{\theta}{}_{\sigma\theta} - \Gamma^{\sigma}{}_{\phi\theta}\Gamma^{\theta}{}_{\sigma\phi}\right), \\ &= a^2\left(\sin^2\theta - \cos^2\theta + \cot\theta\sin\theta\cos\theta\right), \\ &= a^2\sin^2\theta. \end{aligned}$$

The Ricci tensor is given by

$$R_{\beta\gamma} = g^{\alpha\delta}R_{\alpha\beta\gamma\delta}$$

which in this case reduces to

$$R_{\beta\gamma} = g^{\theta\theta}R_{\theta\beta\gamma\theta} + g^{\phi\phi}R_{\phi\beta\gamma\phi}$$

Since the Ricci tensor is symmetric there are only 3 independent components in 2D which are

$$\begin{aligned} R_{\theta\theta} &= g^{\theta\theta}R_{\theta\theta\theta\theta} + g^{\phi\phi}R_{\phi\theta\theta\phi} \\ &= -g^{\phi\phi}R_{\theta\phi\theta\phi} \\ &= -1. \end{aligned}$$

Next

$$R_{\theta\phi} = g^{\theta\theta}R_{\theta\theta\phi\theta} + g^{\phi\phi}R_{\phi\theta\phi\phi} = 0,$$

and finally

$$R_{\phi\phi} = g^{\theta\theta}R_{\theta\phi\phi\theta} + g^{\phi\phi}R_{\phi\phi\phi\phi} = -\sin^2\theta.$$

Thus the Ricci scalar is

$$\begin{aligned} R &= g^{\alpha\beta}R_{\alpha\beta} \\ &= g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} \\ &= -\frac{2}{a^2} \end{aligned}$$

5.3. When developing a model of the Universe, Einstein added an extra term to the field equations so that they read

$$R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta} + \Lambda g^{\alpha\beta} = kT^{\alpha\beta}$$

where Λ is the "cosmological constant" and $k = -8\pi G/c^4$.

(a) Prove that these equations still satisfy the condition ${T^{\alpha\beta}}_{;\alpha} = 0$.

Taking the covariant derivative and contracting its index with α

$$R^{\alpha\beta}_{;\alpha} - \frac{1}{2}R_{,\alpha}g^{\alpha\beta} - \frac{1}{2}R{g^{\alpha\beta}}_{;\alpha} + \Lambda {g^{\alpha\beta}}_{;\alpha} = k{T^{\alpha\beta}}_{;\alpha}$$

where the Leibniz rule for covariant derivatives has been used, and also we have set $R_{;\alpha} = R_{,\alpha}$ since R is scalar. The covariant derivative of the metric is zero, and so

$$R^{\alpha\beta}_{;\alpha} - \frac{1}{2}R_{,\alpha}g^{\alpha\beta} = k{T^{\alpha\beta}}_{;\alpha}$$

As shown in the lectures, the left-hand side is zero by design, and so the term in Λ makes no difference to ${T^{\alpha\beta}}_{;\alpha} = 0$, essentially because ${g^{\alpha\beta}}_{;\alpha} = 0$.

(b) Show that the Ricci scalar $R = -kT + 4\Lambda$ where, as in lectures, $T = g_{\alpha\beta}T^{\alpha\beta}$.

Contracting the field equations on α and β:

$$g_{\alpha\beta}R^{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}g^{\alpha\beta} + \Lambda g_{\alpha\beta}g^{\alpha\beta} = kg_{\alpha\beta}T^{\alpha\beta}$$

Now $g_{\alpha\beta}g^{\alpha\beta} = \delta^{\alpha}_{\alpha} = 4$, so

$$R - 2R + 4\Lambda = kT,$$

so

$$R = -kT + 4\Lambda.$$

QED.

(c) Hence show that

$$R^{\alpha\beta} = k\left(T^{\alpha\beta} - \frac{1}{2}Tg^{\alpha\beta}\right) + \Lambda g^{\alpha\beta}$$

Replacing the Ricci scalar in the field equations using the result above gives

$$R^{\alpha\beta} - \frac{1}{2}\left(-kT + 4\Lambda\right)g^{\alpha\beta} + \Lambda g^{\alpha\beta} = kT^{\alpha\beta}$$

after which a simple re-arrangement leads to the result here.

(d) Therefore, by modifying the Newtonian limit calculation used in lectures to derive the value of k, show that in the limit of slow motion and weak fields, the 00 component of the field equations becomes

$$\nabla^2\phi = 4\pi G\rho - \Lambda c^2.$$

If we follow the derivation from lectures, then from the 00*-cpt we derived*

$$-\frac{1}{c^2}\nabla^2\phi = k\left(\rho c^2 - \frac{1}{2}\rho c^2\right).$$

This just need $\Lambda g^{00} = \Lambda$ *(weak field) added to the RHS. This immediately leads to the new form of the Newtonian limit.*

(e) Show then that the Newtonian potential at distance r from a spherically symmetric mass M is

$$\phi = -\frac{GM}{r} - \frac{\Lambda c^2 r^2}{6}.$$

From Gauss' theorem,

$$\int \nabla\cdot(\nabla\phi)\,dV = \oint \nabla\phi\cdot\hat{\mathbf{n}}\,dA,$$

where $\hat{\mathbf{n}}$ *is a unit outward pointing normal vector for each element of area* dA *covering a closed surface. Taking a sphere of radius* r *centred on a point of spherical symmetry,*

$$\oint \nabla\phi\cdot\hat{\mathbf{n}}\,dA = 4\pi r^2\frac{d\phi}{dr},$$

since $\nabla\phi$ *must everywhere point radially in such a case. Therefore*

$$4\pi r^2\frac{d\phi}{dr} = \int\left(4\pi G\rho - \Lambda c^2\right)\,dV = 4\pi GM - \Lambda c^2\frac{4\pi r^3}{3},$$

so

$$\frac{d\phi}{dr} = \frac{GM}{r^2} - \frac{1}{3}\Lambda c^2 r.$$

Integrating

$$\phi = -\frac{GM}{r} - \frac{1}{6}\Lambda c^2 r^2,$$

ignoring the constant of integration.

(f) What would be the physical effect of the cosmological constant term?

The gravitational field $\mathbf{g} = -\nabla\phi$*, or radially*

$$g = -\frac{GM}{r^2} + \frac{1}{3}\Lambda c^2 r.$$

Thus the cosmological constant acts as a repulsive *force that gets stronger with radius.*

What effect would it have upon the orbital periods of the planets?

The outer planets would need to move more slowly than expected to balance the gravitational pull of the Sun and so would have longer periods than the standard $P^2 \propto a^3$ *scaling from Earth's orbital period.*

(g) In 1997 evidence was found in observations of supernovae that indicates that the cosmological constant is not zero, but has a value of $\Lambda = 1.2 \times 10^{-52}\,\mathrm{m}^{-2}$. Estimate whether this would have observable effects upon the orbits of the planets.

We can get an estimate of the magnitude of the cosmological term by comparing it to GM/r^2. To give it a chance, we will compare it in the outer solar system, 50 AU *from the Sun:*

$$\frac{\Lambda c^2 r}{3GM/r^2} = \frac{\Lambda c^2 r^3}{3GM} = \frac{1.2 \times 10^{-52} \times (3 \times 10^8)^2 \times (50 \times 1.5 \times 10^{11})^3}{3 \times 6.67 \times 10^{-11} \times 2 \times 10^{30}} = 1.1 \times 10^{-17}.$$

The cosmological constant is negligible on Solar system scales.

Note that an unexplained deviation from Newton's law of gravity has been observed in the Pioneer *probes which were tracked for many years after their launch in the early 1970's. This amounts to a constant extra acceleration* towards the Sun *of $a = (8.74 \pm 1.33) \times 10^{-10}\,\mathrm{m\,s}^{-2}$. It is not yet known whether this is some mundane problem or whether it requires new physics. The calculations above make it clear that it is not the result of the cosmological constant which has the wrong sign and is far too feeble.*

5.4. Consider the term $Rg^{\alpha\beta}$ where the Ricci scalar $R = R^{\alpha\beta}g_{\alpha\beta}$. Therefore

$$Rg^{\alpha\beta} = R^{\alpha\beta}g_{\alpha\beta}g^{\alpha\beta},$$

but we know that

$$g_{\alpha\beta}g^{\alpha\beta} = 4,$$

therefore

$$Rg^{\alpha\beta} = 4R^{\alpha\beta}.$$

What is wrong with these statements?

The line

$$Rg^{\alpha\beta} = R^{\alpha\beta}g_{\alpha\beta}g^{\alpha\beta},$$

was a dangerous error comitted by a fair number in the 2011 exam. You should never have more than one identical contravariant (up) or covariant (down) index. The first two symbols on the right contract with each other and are unrelated to the final symbol. The line should have been written

$$Rg^{\alpha\beta} = R^{\mu\nu}g_{\mu\nu}g^{\alpha\beta},$$

whereby the final "simplification" is seen to be entirely wrong.

5.5. Show that, in the absence of a cosmological constant, the field equations in free-space can be written as:

$$R^{\alpha\beta} = 0.$$

We can say that $T^{\alpha\beta} = 0$, so

$$R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta} = 0.$$

However contracting with $g_{\alpha\beta}$ gives

$$R - \frac{1}{2}Rg_{\alpha\beta}g^{\alpha\beta} = -R = 0,$$

and so the equations reduce to

$$R^{\alpha\beta} = 0.$$

QED.

Does this imply that free-space must be flat?

No, it does not. If it did, one could have no gravitational fields in free-space and the planets would all fly off in straight lines from the Sun. The Ricci tensor $R^{\alpha\beta}$ is not *the Riemann curvature tensor, $R_{\alpha\beta\gamma\delta}$. If the latter is zero, we are in flat space, but it has more degrees of freedom (20) than the Ricci tensor (10), so while $R_{\alpha\beta\gamma\delta} = 0$ implies that $R^{\alpha\beta} = 0$, the reverse statement is not true.*

5.6. * Starting from the connection coefficients of part (a) of Q**4.8**, show that the non-zero coefficients of the Ricci tensor for a spherically symmetric spacetime are:

$$\begin{aligned} R_{tt} &= -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB}, \\ R_{rr} &= \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB}, \\ R_{\theta\theta} &= \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right), \\ R_{\phi\phi} &= R_{\theta\theta}\sin^2\theta, \end{aligned}$$

where the dashes and double-dashes indicates first and second derivatives with respect to r.

Set equal to zero (see previous question), these are the equations that lead to the Schwarzschild metric with $A(r) = c^2(1 - 2GM/c^2r)$, $B(r) = (1 - 2GM/c^2r)^{-1}$.

The (non-zero) connection coefficients are $\Gamma^t{}_{tr} = A'/2A$, $\Gamma^r{}_{tt} = A'/2B$, $\Gamma^r{}_{rr} = B'/2B$, $\Gamma^r{}_{\theta\theta} = -r/B$, $\Gamma^r{}_{\phi\phi} = -(r\sin^2\theta)/B$, $\Gamma^\theta{}_{r\theta} = 1/r$, $\Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta$, $\Gamma^\phi{}_{r\phi} = 1/r$, $\Gamma^\phi{}_{\theta\phi} = \cot\theta$. The general formula for the Ricci tensor is

$$R_{\alpha\beta} = \Gamma^\rho{}_{\alpha\rho,\beta} - \Gamma^\rho{}_{\alpha\beta,\rho} + \Gamma^\rho{}_{\alpha\sigma}\Gamma^\sigma{}_{\rho\beta} - \Gamma^\rho{}_{\alpha\beta}\Gamma^\sigma{}_{\rho\sigma}.$$

Thus starting with R_{tt} we have, upon setting $\alpha = t$ and $\beta =$,

$$R_{tt} = \Gamma^\rho{}_{t\rho,t} - \Gamma^\rho{}_{tt,\rho} + \Gamma^\rho{}_{t\sigma}\Gamma^\sigma{}_{\rho t} - \Gamma^\rho{}_{tt}\Gamma^\sigma{}_{\rho\sigma}.$$

The first term drops out because the metric is static and substituting in the connection (remembering that all partial derivatives wrt r become normal derivatives) we get

$$\begin{aligned} R_{tt} &= -\frac{d}{dr}\left(\frac{A'}{2B}\right) + \frac{A'}{2A}\frac{A'}{2B} + \frac{A'}{2B}\frac{A'}{2A} - \frac{A'}{2B}\left(\frac{A'}{2A} + \frac{B'}{2B} + \frac{1}{r} + \frac{1}{r}\right) \\ &= -\frac{A''}{2B} + \frac{A'B'}{2B^2} + \frac{A'}{2B}\left(\frac{A'}{A} - \frac{A'}{2A} - \frac{B'}{2B} - \frac{2}{r}\right), \\ &= -\frac{A''}{2B} + \frac{A'}{2B}\left(\frac{A'}{2A} + \frac{B'}{2B} - \frac{2}{r}\right), \\ &= -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB}. \end{aligned}$$

The other terms follow in a similar manner.

5.7. A line parameterised by λ and with tangent vector $\vec{U}$ obeys the relation

$$\nabla_{\vec{U}}\vec{U} = f\vec{U},$$

where $f = f(\lambda)$ is a function of λ.

(a) Show that, by parameterising the line in terms of a new parameter μ such that

$$\frac{d^2\mu}{d\mu d\lambda} = f,$$

the line is a geodesic.

In component form the relation is

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = f\frac{dx^\alpha}{d\lambda}.$$

The switch to $\mu = \mu(\lambda)$ can be made using

$$\frac{d}{d\lambda} = \frac{d\mu}{d\lambda}\frac{d}{d\mu}.$$

Setting the derivative $d\mu/d\lambda = \mu'$ and applying this relation then gives

$$\mu'\frac{d}{d\mu}\left(\mu'\frac{dx^\alpha}{d\mu}\right) + \Gamma^\alpha{}_{\beta\gamma}(\mu')^2\frac{dx^\beta}{d\mu}\frac{dx^\gamma}{d\mu} = f\mu'\frac{dx^\alpha}{d\mu}.$$

Taking the derivative through the first term leaves

$$(\mu')^2\left(\frac{d^2x^\alpha}{d\mu^2} + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\mu}\frac{dx^\gamma}{d\mu}\right) = \mu'\left(f - \frac{d\mu'}{d\mu}\right)\frac{dx^\alpha}{d\mu}.$$

Therefore if

$$\frac{d\mu'}{d\mu} = \frac{d^2\mu}{d\mu d\lambda} = f,$$

the equation reduces to the standard geodesic equation $\nabla_{\vec{U}'}\vec{U}' = 0$, where $\vec{U}'$ is the tangent vector of the line when parameterised by μ. The line is therefore a geodesic.

(b) Hence show that if f is constant then

$$\mu = Ae^{f\lambda} + B,$$

where A and B are constants.

$$\frac{d^2\mu}{d\mu d\lambda} = \frac{d\lambda}{d\mu}\frac{d}{d\lambda}\left(\frac{d\mu}{d\lambda}\right) = \frac{1}{\mu'}\frac{d\mu'}{d\lambda} = \frac{d\ln(\mu')}{d\lambda},$$

therefore the condition becomes

$$\frac{d\ln(\mu')}{d\lambda} = f.$$

Integrating

$$\ln(\mu') = f\lambda + k,$$

where k is a constant, so

$$\frac{d\mu}{d\lambda} = \exp(f\lambda + k).$$

Integrating for a second time then

$$\mu = f^{-1}\exp(f\lambda + k) + k',$$

where k' is another constant. This is of the form

$$\mu = Ae^{f\lambda} + B,$$

as given in the question.

5.8. Show that the worldline which maximises the integral $\int L\,d\lambda$, where $L = g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta$, and dots denote derivatives with respect to λ, satisfies the relations

$$\ddot{x}^\alpha + \Gamma^\alpha{}_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = 0,$$

where the connection is given by the Levi-Civita relation.

Maximisation or minimisation of the integral leads to the Euler-Lagrange equations

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial\dot{x}^\alpha}\right) - \frac{\partial L}{\partial x^\alpha} = 0,$$

and setting $L = g_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma$, one immediately obtains

$$\frac{d}{d\lambda}\left(g_{\alpha\gamma}\dot{x}^\gamma + g_{\beta\alpha}\dot{x}^\beta\right) - g_{\beta\gamma,\alpha}\dot{x}^\beta\dot{x}^\gamma = 0.$$

The chain rule gives

$$\frac{d}{d\lambda} = \frac{dx^\delta}{d\lambda}\frac{\partial}{\partial x^\delta} = \dot{x}^\delta\frac{\partial}{\partial x^\delta}.$$

Applying this to the term in brackets gives

$$g_{\alpha\gamma}\ddot{x}^\gamma + g_{\beta\alpha}\ddot{x}^\beta + g_{\alpha\gamma,\delta}\dot{x}^\delta\dot{x}^\gamma + g_{\beta\alpha,\delta}\dot{x}^\delta\dot{x}^\alpha - g_{\beta\gamma,\alpha}\dot{x}^\beta\dot{x}^\gamma = 0.$$

Using the symmetry of the metric and relabelling dummy indices appropriately, this can be written as

$$2g_{\alpha\gamma}\ddot{x}^\gamma + \left(g_{\alpha\gamma,\beta} + g_{\beta\alpha,\gamma} - g_{\beta\gamma,\alpha}\right)\dot{x}^\beta\dot{x}^\gamma = 0.$$

Multiplying by $g^{\sigma\alpha}$ and contracting on α

$$2\delta^\sigma_\gamma\ddot{x}^\gamma + g^{\sigma\alpha}\left(g_{\alpha\gamma,\beta} + g_{\beta\alpha,\gamma} - g_{\beta\gamma,\alpha}\right)\dot{x}^\beta\dot{x}^\gamma = 0.$$

Finally re-labelling α to δ and σ to α, and dividing by 2:

$$\ddot{x}^\alpha + \frac{1}{2}g^{\alpha\delta}\left(g_{\delta\gamma,\beta} + g_{\beta\delta,\gamma} - g_{\beta\gamma,\delta}\right)\dot{x}^\beta\dot{x}^\gamma = 0.$$

Recognising the Levi-Civita connection we finally can write

$$\ddot{x}^\alpha + \Gamma^\alpha{}_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = 0.$$

Note that these are only the correct equations of motion if λ is affine. Using $L = \sqrt{g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}$ gives the right equations for any λ, but they only reduce to the simple form here when λ is affine ($dL/d\lambda = 0$).

5.9. * *Linearised GR:* when gravitational fields are weak, one can find coordinates throughout spacetime for which $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ with $|h_{\alpha\beta}| \ll 1$.

(a) Show that to first order in h, the Ricci tensor can be written as

$$R_{\alpha\beta} = \frac{1}{2}\eta^{\gamma\delta}\left(h_{\delta\gamma,\alpha\beta} + h_{\alpha\beta,\delta\gamma} - h_{\alpha\gamma,\delta\beta} - h_{\delta\beta,\alpha\gamma}\right).$$

Starting from the Riemann tensor

$$R^\gamma{}_{\alpha\beta\delta} = \Gamma^\gamma{}_{\alpha\delta,\beta} - \Gamma^\gamma{}_{\alpha\beta,\delta} + \Gamma^\sigma{}_{\alpha\delta}\Gamma^\gamma{}_{\sigma\beta} - \Gamma^\sigma{}_{\alpha\beta}\Gamma^\gamma{}_{\sigma\delta}.$$

The last two terms are second order in h and can immediately be neglected. From the Levi-Civita formula, the first term on the right-hand side is

$$\Gamma^\gamma{}_{\alpha\delta,\beta} = \left[\frac{1}{2}g^{\gamma\sigma}\left(g_{\sigma\delta,\alpha} + g_{\alpha\sigma,\delta} - g_{\alpha\delta,\sigma}\right)\right]_{,\beta}.$$

All the derivative terms are first-order in h, and therefore the metric outside the bracket can be taken to be the Minkowski metric $\eta^{\gamma\sigma}$ and we find

$$\Gamma^\gamma{}_{\alpha\delta,\beta} = \frac{1}{2}\eta^{\gamma\sigma}\left(h_{\sigma\delta,\alpha\beta} + h_{\alpha\sigma,\delta\beta} - h_{\alpha\delta,\sigma\beta}\right).$$

Similarly, the second term gives

$$\Gamma^\gamma{}_{\alpha\beta,\delta} = \frac{1}{2}\eta^{\gamma\sigma}\left(h_{\sigma\beta,\alpha\delta} + h_{\alpha\sigma,\beta\delta} - h_{\alpha\beta,\sigma\delta}\right).$$

Taking the difference, using the commutativity of partial derivatives, we are left with

$$R^{\gamma}{}_{\alpha\beta\delta} = \frac{1}{2}\eta^{\gamma\sigma}\left(h_{\sigma\delta,\alpha\beta} + h_{\alpha\beta,\sigma\delta} - h_{\alpha\delta,\sigma\beta} - h_{\sigma\beta,\alpha\delta}\right).$$

Contracting on γ and δ (i.e. change δ to γ, and then re-labelling the dummy index σ to δ gives the final expression.

(b) Hence show that to first-order in h, the Ricci scalar is given by

$$R = \Box h - h^{\alpha\beta}{}_{,\alpha\beta}\,,$$

where h is the trace, i,e $h = \eta^{\alpha\beta}h_{\alpha\beta}$, $\Box$ is the d'Alembertian

$$\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2,$$

and index raising and lowering involves η rather than g.

Contracting the formula for the Ricci tensor

$$R = \frac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\delta}\left(h_{\delta\gamma,\alpha\beta} + h_{\alpha\beta,\delta\gamma} - h_{\alpha\gamma,\delta\beta} - h_{\delta\beta,\alpha\gamma}\right),$$

where again we can write η rather than g given that we are only retaining first-order terms. The first term can be written as

$$\begin{aligned}
\frac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\delta}h_{\delta\gamma,\alpha\beta} &= \frac{1}{2}\eta^{\alpha\beta}\eta^{\gamma\delta}\partial_\beta\partial_\alpha h_\delta\gamma,\\
&= \frac{1}{2}\eta^{\alpha\beta}\partial_\beta\partial_\alpha\eta^{\delta\gamma}h_\delta\gamma,\\
&= \frac{1}{2}\Box h.
\end{aligned}$$

The second term gives the same quantity. The third and fourth terms can both be regarded as two index rasing operations applied to the last two indices, and after some re-labelling, the answer emerges.

(c) Finally, show that the field equations can be written

$$h_{,\alpha\beta} + \Box h_{\alpha\beta} - \eta^{\gamma\delta}\left(h_{\alpha\gamma,\delta\beta} + h_{\delta\beta,\alpha\gamma}\right) - \left(\Box h - h^{\sigma\rho}{}_{,\sigma\rho}\right)\eta_{\alpha\beta} = -\frac{16\pi G}{c^4}T_{\alpha\beta}.$$

(This equation, which with a careful choice of coordinates can be greatly simplified, is the starting point for the theory of gravitational waves.)

The field equations are

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = -\frac{8\pi G}{c^4}T_{\alpha\beta}.$$

It is then just a case of putting the previous two formulae in, setting the $g_{\alpha\beta} = \eta_{\alpha\beta}$ to first order, multiplying the whole thing by 2 and spotting that the first two terms of the Ricci tensor can be re-written slightly.

5.10. By working in inertial coordinates, prove the *Bianchi identity*, given, but not proved, in the handouts:

$$R_{\alpha\beta\gamma\delta;\mu} + R_{\alpha\beta\mu\gamma;\delta} + R_{\alpha\beta\delta\mu;\gamma} = 0.$$

[Note the way the last three indices are cyclicly permuted.]

This question shows the power of inertial coordinates which simplify the work enormously. In inertial coordinates first-order derivatives of the metric $g_{\alpha\beta,\gamma}$ and therefore connection coefficients ${\Gamma^\alpha}_{\beta\gamma}$ are all zero. Therefore consider $R_{\alpha\beta\gamma\delta;\mu}$. Fully expanded this has a correction term involving the connection for each of the first four indices. In inertial coordinates however these all disappear and we are left with

$$R_{\alpha\beta\gamma\delta;\mu} = R_{\alpha\beta\gamma\delta,\mu},$$

an enormous simplification. The Riemann tensor has four terms, two of which involve products of the connection. When we take the derivatives of these using the product rule, the resulting terms will have the connection multiplying a derivative of the connection. However again inertial coordinates means that the connection is zero, so these terms drop out as well. Remembering too that the metric derivatives are zero we are left with

$$R_{\alpha\beta\gamma\delta;\mu} = g_{\alpha\rho}\left({\Gamma^\rho}_{\beta\delta,\gamma\mu} - {\Gamma^\rho}_{\beta\gamma,\delta\mu}\right).$$

This is a huge simplification but the price we have paid is that this is not a tensor relation: it only holds in inertial coordinates. Interchanging indices to get the other two terms gives us the three relations (the first simply a repetition of the equation above):

$$\begin{aligned}
R_{\alpha\beta\gamma\delta;\mu} &= g_{\alpha\rho}\left({\Gamma^\rho}_{\beta\delta,\gamma\mu} - {\Gamma^\rho}_{\beta\gamma,\delta\mu}\right),\\
R_{\alpha\beta\mu\gamma;\delta} &= g_{\alpha\rho}\left({\Gamma^\rho}_{\beta\gamma,\mu\delta} - {\Gamma^\rho}_{\beta\mu,\gamma\delta}\right),\\
R_{\alpha\beta\delta\mu;\gamma} &= g_{\alpha\rho}\left({\Gamma^\rho}_{\beta\mu,\delta\gamma} - {\Gamma^\rho}_{\beta\delta,\mu\gamma}\right).
\end{aligned}$$

If these are added then using the symmetry on the two lower indices of the connection and the commutativity of partial differentiation (symmetry in the two indices after the commas), it is easily seen that all terms cancel and thus

$$R_{\alpha\beta\gamma\delta;\mu} + R_{\alpha\beta\mu\gamma;\delta} + R_{\alpha\beta\delta\mu;\gamma} = 0.$$

Now this <u>is</u> a tensor relation because the LHS is clearly a tensor and 0 represents the most trivial tensor of all. So this is a covariant relation that holds in all coordinates. Very neat you have to admit!

5.11. Prove that the Ricci tensor is symmetric.

The Ricci tensor is defined by

$$R_{\alpha\beta} = g^{\mu\nu} R_{\mu\alpha\beta\nu}.$$

The symmetry/anti-symmetry relations of the Riemann tensor are

$$\begin{aligned}
R_{\alpha\beta\gamma\delta} &= -R_{\beta\alpha\gamma\delta},\\
R_{\alpha\beta\gamma\delta} &= -R_{\alpha\beta\delta\gamma},\\
R_{\alpha\beta\gamma\delta} &= R_{\gamma\delta\alpha\beta}.
\end{aligned}$$

Applying these and also using the symmetry of the metric and suitable component re-labelling shows that

$$\begin{aligned} R_{\alpha\beta} &= g^{\mu\nu}R_{\mu\alpha\beta\nu}, \\ &= g^{\mu\nu}R_{\beta\nu\mu\alpha}, \\ &= g^{\mu\nu}R_{\nu\beta\alpha\mu}, \\ &= g^{\nu\mu}R_{\nu\beta\alpha\mu}, \\ &= g^{\mu\nu}R_{\mu\beta\alpha\nu}, \\ &= R_{\beta\alpha}, \end{aligned}$$

QED.

5.12. Show that in one-dimension there is no Riemann tensor while in two dimensions it has only one independent component.

The only component is R_{1111} *for a coordinate labelled with index* 1. *The anti-symmetry in the last two indices proves that this must* $= 0$, *QED.*

In 2 dimensions there are potentially $2^4 = 16$ *components which have indices* 1111, 1112, 1121, 1122, 1211, 1212, 1221, 1222, 2111, 2112, 2121, 2122, 2211, 2212, 2221, 2222. *The anti-symmetry relations*

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= -R_{\beta\alpha\gamma\delta}, \\ R_{\alpha\beta\gamma\delta} &= -R_{\alpha\beta\delta\gamma}, \end{aligned}$$

mean that we can eliminate all terms with a repeated index in the first or last pair of indices and we are left with just four non-zero terms with indices 1212, 1221, 2112, 2121. *However the above two relations show all these terms are the same, give or take a sign, so there is only one independent component. This single number is effectively what Carl Friedrich Gauss came across in his "Theorema Egregium" (Remarkable Theorem) for the curvature of a 2D surface. He found it remarkable that his formula only involved lengths within the surface despite his setting up the problem from a 3D viewpoint. It took the work of Riemann 30 years later to extend this work to more than 2 dimensions.*

5.13. * Show that the number of independent components of the Riemann tensor in N dimensions is given by

$$\frac{1}{12}N^2(N^2-1).$$

There are a possible N^4 *combinations of 4 indices (256 for* $N = 4$*), but the number of independent values is severely limited by the symmetry relations:*

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= -R_{\beta\alpha\gamma\delta}, \\ R_{\alpha\beta\gamma\delta} &= -R_{\alpha\beta\delta\gamma}, \\ R_{\alpha\beta\gamma\delta} &= R_{\gamma\delta\alpha\beta}, \end{aligned}$$

and

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0.$$

The first two show that Riemann tensor in antisymmetric in the first and last pairs of indices and thus there are $N' = N(N-1)/2$ *different combinations of each pair, giving a potential* $(N')^2$ *components, 36 for* $N = 4$*. However the third relation shows symmetry between the first and last index pairs and thus there are*

$$\frac{1}{2}N'(N'+1) = \frac{1}{2}\left(\frac{N(N-1)}{2}\right)\left(\frac{N(N-1)}{2}+1\right) = \frac{1}{8}N(N-1)(N^2-N+2),$$

possible values, which gives 21 *for* $N = 4$*. We finally need to include the fourth constraint. One has to be careful not to overcount the constraints from this given the other symmetries. For instance, consider the case of* $\beta = \alpha$*, then*

$$R_{\alpha\alpha\gamma\delta} + R_{\alpha\gamma\delta\alpha} + R_{\alpha\delta\alpha\gamma} = 0.$$

Anti-symmetry on the first pair of indices removes the first term, and anti-symmetry on the second pair of indices allows us to reverse the indices on the second pair of the second term to write

$$R_{\alpha\delta\alpha\gamma} = R_{\alpha\gamma\alpha\delta}.$$

But this is just the third of the symmetry relations, so we have learned nothing new. Therefore we must require $\alpha \neq \beta$*. Similar reasoning shows that all of the indices must be different. The symmetries mean that the precise order of the indices is immaterial, so the number of constraints is the number of ways of picking 4 objects from* N*, regardless of order:*

$$\frac{N!}{4!(N-4)!}.$$

Thus the final number of independent coefficients is given by

$$\frac{1}{8}N(N-1)(N^2-N+2) - \frac{1}{24}N(N-1)(N-2)(N-3) = \frac{1}{24}N(N-1)(3N^2-3N+6-N^2+5N-6).$$

We are left with

$$\frac{1}{24}N(N-1)(2N^2+2N) = \frac{1}{12}N^2(N^2-1),$$

QED. For $N = 4$*, this gives 20 as quoted in lectures.*

6.1. The non-zero coefficients of Einstein's field equations in empty space for a spherically symmetric metric of the form

$$ds^2 = A(r)\,dt^2 - B(r)\,dr^2 - r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right),$$

are as follows:

$$R_{tt} = -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB} = 0, \tag{9}$$

$$R_{rr} = \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB} = 0, \tag{10}$$

$$R_{\theta\theta} = \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) = 0. \tag{11}$$

(The $R_{\phi\phi}$ component carries the same information as the $R_{\theta\theta}$ component.)

(a) By adding $B\times$ Eq. 9 to $A\times$ Eq. 10 show that $AB = \alpha$, a constant.

Carrying out the manipulation indicated, the first two terms of each equations cancel leaving

$$-\frac{BA'}{rB} - \frac{AB'}{rB} = 0,$$

or

$$\frac{d}{dr}(AB) = 0,$$

hence $AB = \alpha$, a constant. QED

(b) Hence use Eq. 11 to show that $d(rA)/dr = \alpha$, and so

$$A(r) = \alpha\left(1 + \frac{k}{r}\right),$$

where k is another integration constant.

Setting $B' = -BA'/A$ and $B = \alpha/A$, then Eq. 11 reads

$$\frac{A}{\alpha} - 1 + \frac{rA}{2\alpha}\left(\frac{A'}{A} + \frac{A'}{A}\right) = 0,$$

or

$$A + rA' = \alpha = \frac{d}{dr}(rA),$$

and finally

$$A = \alpha\left(1 + \frac{k}{r}\right),$$

where k is another integration constant.

(c) Finally, by considering the weak field limit, justify the Schwarzschild metric.

In weak fields the dt^2 coefficient becomes

$$A(r) = c^2 \left(1 + \frac{2\phi}{c^2}\right),$$

and setting the Newtonian potential $\phi = -GM/r$, we deduce $\alpha = c^2$, $k = -2GM/c^2$.

6.2. With the cosmological constant, the field equations in empty-space become

$$R^{\alpha\beta} = \Lambda g^{\alpha\beta}.$$

Write down the modified versions of Eqs 9, 10 and 11 from Q**6.1**, and repeat the working to show that the metric becomes

$$ds^2 = c^2 \left(1 - \frac{2GM}{c^2 r} - \frac{\Lambda r^2}{3}\right) dt^2 - \left(1 - \frac{2GM}{c^2 r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right).$$

The equations become

$$\begin{aligned}
-\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB} &= \Lambda g_{tt} = \Lambda A, \\
\frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB} &= \Lambda g_{rr} = -\Lambda B, \\
\frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) &= \Lambda g_{\theta\theta} = -\Lambda r^2.
\end{aligned}$$

The same procedure as before then gives

$$\frac{d}{dr}(AB) = 0,$$

again, so $AB = \alpha$ still. Then the second part gives

$$\frac{d}{dr}(rA) = \alpha\left(1 - \Lambda r^2\right),$$

which on integration yields

$$rA = \alpha\left(r + k - \frac{\Lambda r^3}{3}\right).$$

As before we know that for large r (but not so large that the cosmological constant matters), $A \to c^2(1 + 2\phi/c^2)$, so as before $\alpha = c^2$ and $k = -2GM/c^2$ and therefore the modified version of the metric follows straightforwardly.

6.3. Person A, stationary at $r = r_A$ from a mass M (Schwarzschild radial coordinates), regularly sends pulses of light in the radial direction to person B who is stationary at $r = r_B > r_A$.

(a) Show that along the path of the light-pulse

$$c\,dt = \frac{dr}{1 - 2GM/c^2 r}.$$

Since the path is radial (and by symmetry it must be radial the whole way from A to B), then $d\theta = d\phi = 0$ and we have

$$ds^2 = c^2 \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2.$$

Since the path is null $ds^2 = 0$, and thus

$$c\,dt = \pm \frac{dr}{1 - 2GM/c^2 r}.$$

Clearly the positive root applies if the light travels outwards.

(b) Show that each pulse take the same coordinate time to travel from A to B.

This is one of those so-obvious-you-miss-it questions. The time is

$$c(t_B - t_A) = \int_{r_A}^{r_B} \frac{dr}{1 - 2GM/c^2 r},$$

which is always the same since it contains no time dependence on the right-hand side.

(c) Hence show that, if A transmits pulses at rate ν_A, then B receives them at rate ν_B where

$$\frac{\nu_B}{\nu_A} = \left(\frac{1 - 2GM/c^2 r_A}{1 - 2GM/c^2 r_B}\right)^{1/2}.$$

The previous result shows that $\Delta t_A = \Delta t_B$ where Δt_A is the coordinate time between pulses sent by A, and Δt_B is the coordinate time between pulses received by B. However for both A and B, $dr = d\theta = d\phi = 0$, so

$$c^2\,d\tau^2 = c^2 \left(1 - \frac{2GM}{c^2 r}\right) dt^2,$$

so

$$\frac{\Delta\tau_A}{\Delta\tau_B} = \frac{\Delta\nu_B}{\Delta\nu_A} = \left(\frac{1 - 2GM/c^2 r_A}{1 - 2GM/c^2 r_B}\right)^{1/2} \frac{\Delta t_A}{\Delta t_B},$$

and given the equality of the coordinate time increments, the result follows.

(d) What is the physical meaning of the equation of the previous part?

As A is closer to the mass M, $\nu_B/\nu_A < 1$, and B must deduce that A's time progresses more slowly than his/her own time. This is gravitational time dilation.

(e) By how much would the readings of two clocks differ after one year, if one was on the surface of Earth while the other was far from the Earth but stationary with respect to it, assuming that they were initially synchronised?

The clock on Earth would tick at

$$\sqrt{1 - 2GM_E/c^2R_E} \approx 1 - \frac{GM_E}{c^2R_E}$$

the rate of the clock in space, so after 1 year it will have fallen behind by

$$\frac{GM_E}{c^2R_E}\Delta t = \frac{gR_E}{c^2}\Delta t = \frac{9.81 \times 6370 \times 10^3}{(2.9979 \times 10^8)^2} \times 365 \times 24 \times 3600 = 0.022\ \text{sec}\,.$$

(f) The spectrum of the surface of a white dwarf of radius $R = 6000\,\text{km}$, mass $M = 1\,\text{M}_\odot$ shows strong absorption at the wavelength of Hα, which has laboratory rest wavelength $\lambda = 656.276\,\text{nm}$.

i. What wavelength would be measured on Earth, assuming that the white dwarf is stationary with respect to Earth?
[You may ignore any gravitational effects due to Earth.]

Setting $r_A = 6000\,km$ *and* $r_B = \infty$,

$$\frac{\nu_B}{\nu_A} = \frac{\lambda_A}{\lambda_B} = \left(1 - 2GM/c^2r_A\right)^{1/2} = \left(1 - \frac{2 \times 6.67 \times 10^{-11} \times 2 \times 10^{30}}{(3 \times 10^8)^2 \times 6 \times 10^6}\right)^{1/2} = 0.99975293,$$

which gives $\lambda_B = 656.438\,\text{nm}$.

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ii. An astronomer takes a spectrum of this white dwarf and mis-interprets the offset wavelength as a Doppler shift. What spurious velocity along the line-of-sight would the astronomer measure?

The first-order Doppler shift equation is

$$\lambda = \lambda_0\left(1 + \frac{v}{c}\right),$$

where v is the component of velocity along the line-of-sight. Hence

$$v = c\left(\frac{\lambda}{\lambda_0} - 1\right) = 75\,\text{km}\,\text{s}^{-1}.$$

(g) Which has a more significant effect upon the clocks in a typical commercial air flight ($v \approx 900\,\mathrm{km\,hr^{-1}}$, $h \approx 10{,}000\,\mathrm{m}$), the speeding up of time from being at a higher gravitational potential than the ground or the slowing down from special relativistic time dilation?

Gravitational time dilation factor $1+gh/c^2$, *SR Lorentz factor* $(1-v^2/c^2)^{1/2} \approx 1-v^2/2c^2$. *For this case*

$$\frac{gh}{c^2} = \frac{9.81 \times 10^4}{(3 \times 10^8)^2} = 1.1 \times 10^{-12},$$

while

$$\frac{v^2}{2c^2} = \frac{(900 \times 10^3/3600)^2}{2 \times (3 \times 10^8)^2} = 3.5 \times 10^{-13},$$

so the gravitational effect will win out overall.

6.4. Obtain an expression for the coordinate time taken for light to travel radially from r_1 to r_2 in Schwarzschild coordinates.

From the question above:

$$\begin{aligned} c(t_2 - t_1) &= \int_{r_1}^{r_2} \frac{dr}{1 - 2GM/c^2 r}, \\ &= \int_{r_1}^{r_2} \frac{r\,dr}{r - 2GM/c^2}, \\ &= \int_{r_1}^{r_2} \left(\frac{r - 2GM/c^2}{r - 2GM/c^2} + \frac{2GM/c^2}{r - 2GM/c^2} \right) dr, \\ &= \left[r + \frac{2GM}{c^2} \ln(r - 2GM/c^2) \right]_{r_1}^{r_2}, \\ &= (r_2 - r_1) + \frac{2GM}{c^2} \ln \frac{r_2 - 2GM/c^2}{r_1 - 2GM/c^2}. \end{aligned}$$

Calculate the coordinate time taken for light to travel from Earth to Mercury (0.38 AU from the Sun) and back to Earth, assuming a purely radial path, and compare with the simple formula $\Delta t = 2\Delta r/c$.

The difference between the two estimates (multiplied by 2 for a 2-way trip) amounts to

$$\frac{4GM}{c^3} \ln \frac{r_2 - 2GM/c^2}{r_1 - 2GM/c^2} \approx \frac{4GM}{c^3} \ln \frac{r_2}{r_1} \approx 19.3 \times 10^{-6}\ \mathrm{sec}\,.$$

In practice this so-called "Shapiro delay" is measured by bouncing radio pulses off Mercury or Venus when they are on the opposite side of the Sun from us so that the radio waves pass through the deep potential close to the Sun and delays of order 200 *microseconds ensure.*

6.5. An observer is stationary at Schwarzschild radial coordinate r from a mass M. Starting from

$$A^\alpha = \frac{DU^\alpha}{D\tau} = \frac{dU^\alpha}{d\tau} + \Gamma^\alpha{}_{\beta\gamma} U^\beta U^\gamma,$$

show that the observer experiences an acceleration a given by

$$a = \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} \frac{GM}{r^2}.$$

In which direction does this acceleration point?

The acceleration experienced by the observer is his or her proper acceleration, a, an invariant which comes from the relation

$$a^2 = -\vec{A} \cdot \vec{A}.$$

Since the observer is stationary, $U^i = 0$ (spatial components of four velocity are zero) and from the invariant $\vec{U} \cdot \vec{U} = c^2$ we obtain

$$g_{tt} \left(U^t\right)^2 = c^2.$$

Since for the Schwarzschild metric, $g_{tt} = c^2(1 - 2\mu/r)$, we get

$$U^t = \left(1 - \frac{2\mu}{r}\right)^{-1/2}.$$

This is time-independent and so $dU^\alpha/d\tau = 0$, and we are left with

$$A^\alpha = \Gamma^\alpha{}_{tt} U^t U^t = \left(1 - \frac{2\mu}{r}\right)^{-1} \Gamma^\alpha{}_{tt}.$$

Using the Levi-Civita equation and using the fact that the Schwarzschild metric is diagonal, we can write

$$\Gamma^\alpha{}_{tt} = \frac{1}{2} g^{\alpha\alpha} \left(g_{\alpha t,t} + g_{t\alpha,t} - g_{tt,\alpha}\right).$$

Since the Schwarzschild metric is time-independent, this further reduces to

$$\Gamma^\alpha{}_{tt} = -\frac{1}{2} g^{\alpha\alpha} g_{tt,\alpha}.$$

Since g_{tt} is a function of r alone, the only non-zero term is

$$\Gamma^r{}_{tt} = -\frac{1}{2} g^{rr} g_{tt,r}.$$

Using $g^{rr} = 1/g_{rr} = -(1 - 2\mu/r)$ and $g_{tt} = c^2(1 - 2\mu/r)$, we get

$$\Gamma^r{}_{tt} = \left(1 - \frac{2\mu}{r}\right) \frac{GM}{r^2}.$$

We are left with

$$A^r = \frac{GM}{r^2},$$

as the only non-zero component. This is a deceptively simple result, that depends upon the choice of coordinates. However, it is clearly an outwardly-directed acceleration. This is the acceleration provided by the floor for instance when you stand. The proper acceleration is thus

$$a^2 = -\vec{A} \cdot \vec{A} = -g_{rr} (A^r)^2 = \left(1 - \frac{2\mu}{r}\right)^{-1} \left(\frac{GM}{r^2}\right)^2 .$$

The equation given follows. Note the following: as one approaches the event horizon $r \to 2\mu$ and it becomes increasingly hard to keep stationary. For $r < 2\mu$ the result is non-physical indicating that our assumptions have broken down. In this case it is the assumption of a "stationary observer": there is no such thing inside the event horizon.

6.6. In empty space $T^{\alpha\beta} = 0$, and the field equations reduce to $R^{\alpha\beta} = 0$. A tensor that is zero in one frame is zero in all frames. Thus there is no curvature in empty space. True or false?

False. The Ricci tensor is a contraction of the full Riemann tensor which need not be zero even if the Ricci tensor is. If this were the case, the planet would move in straight lines.

6.7. No general method has been established for proving whether a given metric has a singularity. A guide is to calculate scalar invariants.

Look up the coefficients of the Riemann tensor for the Schwarzschild geometry and hence calculate the value of the scalar

$$R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}.$$

Show that it is finite at the Schwarzschild radius but singular at $r = 0$.

TBD

Why is the Ricci scalar not a useful guide in this case?

Because it is zero.

6.8. How large would a sphere of material with the same density as air have to be for its Schwarzschild radius to exceed its own radius? Compare the radius you estimate with the size of the solar system.

One requires that

$$R = \frac{2GM}{c^2} = \frac{8\pi G\rho R^3}{3c^2},$$

so

$$R = \left(\frac{3c^2}{4\pi G\rho}\right)^{1/2} .$$

Setting $\rho = 1\,\mathrm{kg\,m^{-3}}$ *gives* $R = 121, AU$*, about 3 times the radius of Pluto's orbit.*
