3.1. A cloud of charges has charge density ρ_0 in a frame in which the charges are stationary. Defining the current density four-vector $\vec{J} = \rho_0 \vec{U}$, where $\vec{U}$ is the four-velocity, one can show that $J^\alpha{}_{,\alpha} = 0$.

(a) Why must $\vec{J}$ be a four-vector?

Because ρ_0 is defined in the IRF, all observers agree on its value and it is therefore a scalar, while $\vec{U}$ is a vector, so their product must be a vector.

(b) Now consider a frame in which the charges move with 3-velocity $\mathbf{v}$. Defining $\rho = \gamma\rho_0$ and $\mathbf{j} = \gamma\rho_0\mathbf{v}$, write out $J^\alpha{}_{,\alpha} = 0$ in full.

We can write $\vec{J} = \rho_0\gamma(c, \mathbf{v}) = (c\rho, \mathbf{j})$, so $J^\alpha{}_{,\alpha} = 0$ becomes

$$\frac{\partial c\rho}{\partial ct} + \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z} = 0,$$

or

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0.$$

The only thing to note is that we can interpret $\gamma\rho_0$ as the charge density because electric charge q is a scalar.

(c) Give a physical interpretation of the resulting expression.

The equation expresses conservation of electric charge. The first term, the rate of change of charge density, is balanced by the second term which is the divergence of the charge flux, aka current density.

3.2. The surface of a cylinder radius R with its axis along the z axis can be labelled by the coordinates ϕ and z of the usual cylindrical coordinates r, ϕ, z.

(a) Write down an expression for the interval/line-element ds^2 in these coordinates.

$$dl^2 = R^2\, d\phi^2 + dz^2.$$

(b) Obtain an expression for the line element in terms of new coordinates x and y where $x = R\phi$ and $y = z$.

Put $d\phi = (1/R)dx$, $dz = dy$ *and we get, trivially,*

$$dl^2 = dx^2 + dy^2.$$

(c) In terms of its "interior" geometry, as represented by the line element, is the surface of the cylinder flat or curved?

It is flat because we have managed to transform the metric into that of Euclidean 2D. Our normal view of the surface of a cylinder has it "embedded" in 3D; Riemannian geometry is concerned with the geometry that 2D beings on the surface could determine by making measurements, unaware of any third dimension. They may be surprised on taking a trip around the cylinder to come back to where they first started and they would probably have to think about circles (1D spaces) to understand how this was possible, but this is global topology not local geometry. GR also tells us about such local geometry but not the global topology. Our Universe could have some of the same features, e.g. a straight line journey could end up at the starting point. We will often use such "embedded" analogies, but note that it is not always possible to embed spaces in those of higher dimension like this.

3.3. In terms of the usual spherical polar angles, θ and ϕ, the line element on the surface of a sphere of radius R can be written

$$ds^2 = R^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right).$$

(a) Defining $r = R \sin\theta$, show that the line element can be re-written in terms of r and θ as

$$ds^2 = \frac{dr^2}{1 - (r/R)^2} + r^2 \, d\phi^2.$$

Taking differentials,

$$dr = R\cos(\theta)\, d\theta,$$

so

$$d\theta = \frac{dr}{R\sqrt{1-(r/R)^2}}.$$

The result immediately follows on substitution into the original metric.

(b) The line element of part (a) exhibits a singularity at $r = R$. Identify which region of the sphere this corresponds to; a similar "coordinate singularity" occurs in the Schwarzschild metric.

$r = R$ *when* $\theta = 90^\circ$, *which corresponds to the equator of the sphere. This is not in any way an odd region, which shows how the choice of coordinates can lead to apparent singular behaviour. The same thing occurs at the Schwarzschild radius in the coordinates of the Schwarzschild metric.*

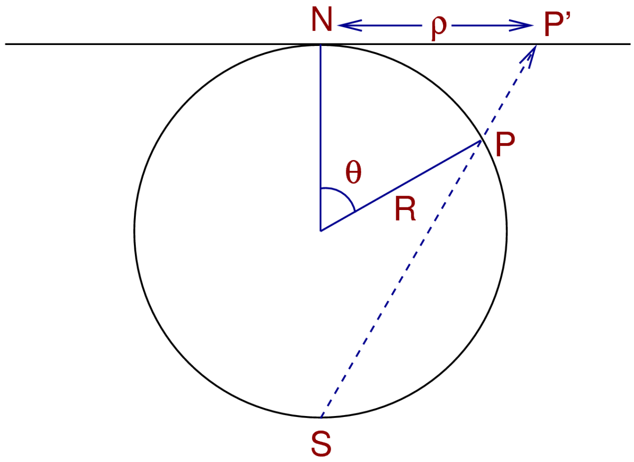

Figure 1: A stereographic projection.

3.4. Figure 1 shows a "stereographic projection" in which a point P on the surface of Earth is projected to P' in a plane tangent at the North pole N along a line running from the South pole S through P. Polar coordinates ρ, ϕ in the tangent plane with N at the origin are then coordinates for the surface of the sphere (except for S).

(a) Show that, in terms of the usual spherical polar angle θ, assuming a z axis running from S to N, ρ is given by

$$\rho = 2R\tan(\theta/2),$$

where R is Earth's radius.

From the geometry of circles, the angle subtended by N and P at S is half the value as seen from the centre of the circle. Since $NP = 2R$, the result follows immediately.

(b) Hence, starting from the spherical interval $ds^2 = R^2\,d\theta^2 + R^2\sin^2\theta\,d\phi^2$, show that in terms of coordinates ρ, ϕ the interval is given by

$$ds^2 = \frac{d\rho^2 + \rho^2\,d\phi^2}{\left(1 + (\rho/2R)^2\right)^2}.$$

Again taking differentials,

$$d\rho = 2R\sec^2(\theta/2)\,\frac{d\theta}{2},$$

therefore

$$d\rho = R\left(1 + (\rho/2R)^2\right)\,d\theta.$$

This relation can be used to substitute for $d\theta$ while

$$\sin\theta = \frac{2\tan(\theta/2)}{1+\tan^2(\theta/2)} = \frac{\rho/R}{1+(\rho/2R)^2}.$$

Using these the metric relation as given can be found [NB exercise 2.4 in Hobson related to this one has an error in it.]

(c) * The stereographic projection preserves the shapes of small regions on the Earth and is thus said to be "conformal". Can you see why this is from the above line element?

The top part of the metric is the flat-space metric of the tangent plane, $d\rho^2 + \rho^2\, d\phi^2$, so the metric of the sphere is simply a scaled version of this. Thus at any point all lengths are simply scaled by the same factor, $1/(1 + (\rho/2R)^2)^2$, regardless of their direction. A small triangle then must scale into a similar triangle, and similarly any small shape which can be built from small triangles will retain its shape.

3.5. * The equations of Newtonian fluid mechanics in Cartesian component form (no distinction between co- and contra-variant indices, all subscripted) for inviscid fluids are

$$\frac{\partial \rho}{\partial t} + \nabla_i(\rho v_i) = 0, \tag{3}$$

and

$$\rho\left(\frac{\partial v_i}{\partial t} + v_j \nabla_j v_i\right) = -\nabla_i p, \tag{4}$$

where $\nabla_i = \partial/\partial x^i$, $i = 1$, 2 or 3, ρ is the density, v_i is a component of the (three-)velocity and p is the pressure. (Summation convention still applies to repeated indices.)

Show from Equations 1 and 2 that

$$\frac{\partial \rho v_i}{\partial t} + \nabla_j t_{ij} = 0, \tag{5}$$

where the tensor t_{ij} is given by

$$t_{ij} = \rho v_i v_j + p\delta_{ij}.$$

Equations 1 and 3 express mass and momentum conservation and are the Newtonian analogues of the SR relations $T^{\alpha\beta}{}_{,\beta} = 0$.

The first term of the second equation can be written

$$\rho\frac{\partial v_i}{\partial t} = \frac{\partial \rho v_i}{\partial t} - v_i\frac{\partial \rho}{\partial t},$$

while the second term becomes

$$\rho v_j \nabla_j v_i = \nabla_j \rho v_j v_i - v_i \nabla_j \rho v_j.$$

The right-hand side thus becomes

$$\frac{\partial \rho v_i}{\partial t} + \nabla_j \rho v_j v_i - v_i\left(\frac{\partial \rho}{\partial t} + \nabla_j \rho v_j\right).$$

The term in brackets is zero by the first (continuity) equation and thus

$$\frac{\partial \rho v_i}{\partial t} + \nabla_j \rho v_j v_i = -\nabla_i p.$$

The right-hand side's term can be written $\nabla_j p\delta ij$ *and therefore*

$$\frac{\partial \rho v_i}{\partial t} + \nabla_j t_{ij} = 0,$$

where the tensor t_{ij} *is as given in the question. This tensor is identical in form to the spatial components of the relativistic stress–energy tensor.*

3.6. The following problem illustrates a subtlety not covered in the lectures: not all bases can be represented by coordinates. The orthogonal unit vectors commonly used in polar and spherical polar coordinates are examples of such *non-coordinate* bases. Other than this problem, all bases in this course are assumed to be coordinate bases as it simplifies several expressions. However, the last part of the question illustrates one of the resulting pitfalls if you extrapolate too simply from familiar results from the past.

Consider a transformation from 2D Cartesian coordinates (x, y) to new coordinates (u, v). Let the transformation of basis vectors be written as

$$\begin{aligned} \vec{e}_u &= L_u{}^x \, \vec{e}_x + L_u{}^y \, \vec{e}_y, \\ \vec{e}_v &= L_v{}^x \, \vec{e}_x + L_v{}^y \, \vec{e}_y. \end{aligned}$$

(a) By comparing this expression with the general formula for the transformation of basis vectors, show that the coefficients above must obey the following conditions:

$$\begin{aligned} \partial_v L_u{}^x &= \partial_u L_v{}^x, \\ \partial_v L_u{}^y &= \partial_u L_v{}^y. \end{aligned}$$

The general transformation of basis vectors under a coordinate transform is given by

$$\vec{e}_{\alpha'} = \frac{\partial x^\beta}{\partial x^{\alpha'}} \vec{e}_\beta.$$

In this case, this can be written as

$$\begin{aligned} \vec{e}_u &= \frac{\partial x}{\partial u} \vec{e}_x + \frac{\partial y}{\partial u} \vec{e}_y, \\ \vec{e}_v &= \frac{\partial x}{\partial v} \vec{e}_x + \frac{\partial y}{\partial v} \vec{e}_y. \end{aligned}$$

Hence $L_u{}^x = \partial x / \partial u$ *etc. Since*

$$\frac{\partial^2 x}{\partial u \partial v} = \frac{\partial^2 x}{\partial v \partial u},$$

and similarly with y *on top, the conditions given follow immediately.*

(b) Show that the usual unit vectors, $\hat{\mathbf{r}}$ and $\hat{\theta}$, which point in the directions of increasing polar coordinates r and θ, *do not* satisfy the above conditions and hence are *not* a coordinate basis.

The transformation in this case is given by the well-known relations

$$\begin{aligned}\hat{\mathbf{r}} &= +\cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}},\\ \hat{\theta} &= -\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}}.\end{aligned}$$

Thus $L_r{}^x = \cos\theta$ *while* $L_\theta{}^x = -\sin\theta$. *Since* $\partial\cos\theta/\partial\theta = -\sin\theta$, *while* $\partial\sin\theta/\partial r = 0$, *it is evident that* $\hat{\mathbf{r}}$ *and* $\hat{\theta}$ *are not a coordinate basis.*

(c) Show that if instead the θ basis vector is taken to be $\vec{e}_\theta = r\hat{\theta}$, then the new basis vectors are a coordinate basis.

Now $L_\theta{}^x = -r\sin\theta$ *and* $\partial(-r\sin\theta)/\partial r = -\sin\theta$ *which does satisfy the required condition, and it is easily shown that the other does as well.*

(d) Hence show that the velocity vector in a polar coordinate basis has components $(v_r, v_\theta/r)$, where v_r and v_θ are the usual speeds along the respective directions.

Since the basis vector in the θ *direction is* r *times the usual unit vector, the component must be the usual speed in the angular direction divided by* r. *More formally we must have*

$$v_\theta = \frac{\vec{V}\cdot\vec{e}_\theta}{\sqrt{\vec{e}_\theta\cdot\vec{e}_\theta}},$$

or since the scalar product is defined by operating the metric tensor on the respective vector arguments:

$$v_\theta = \frac{V^\theta g_{\theta\theta}}{\sqrt{g_{\theta\theta}}}.$$

Since $g_{\theta\theta} = r^2$ *in polar coordinates, then the* θ *component of the velocity is given by* $V^\theta = v_\theta/r$, *as specified.*

4.1. Consider a 2D space of constant curvature which can be pictured in 3D as the surface of a table-tennis ball. Would a 2D being confined to this space have any equivalents of our concepts of "inside" and "outside" surfaces?

No, it would not. No length measurements distinguish an outside and inside surface.

4.2. Can a 1D space be curved?

No, it can't. One can always find a coordinate x^0 *such that the metric of a 1D space can be written* $dl^2 = (dx^0)^2$*, which is Euclidean.*

4.3. Write out the covariant derivative $T_{\alpha\beta;\gamma}$ in full, where $T_{\alpha\beta}$ is an arbitrary tensor.

$$T_{\alpha\beta;\gamma} = T_{\alpha\beta,\gamma} - \Gamma^{\sigma}{}_{\alpha\gamma} T_{\sigma\beta} - \Gamma^{\sigma}{}_{\beta\gamma} T_{\alpha\sigma}.$$

4.4. Calculate the Christoffel symbols for the surface of a sphere of radius R using the usual spherical polar angles (θ, ϕ) as coordinates.

The interval is given by

$$dl^2 = R^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right).$$

Thus the metric components are $g_{\theta\theta} = R^2$, $g_{\phi\phi} = R^2 \sin^2\theta$ *and* $g_{\theta\phi} = 0$. *This is diagonal, so* $g^{\theta\theta} = R^{-2}$ *and* $g^{\phi\phi} = R^{-2} \sin^{-2}\theta$. *We now apply the Levi-Civita equation:*

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(g_{\delta\gamma,\beta} + g_{\beta\delta,\gamma} - g_{\beta\gamma,\delta} \right).$$

We have 6 independent components to calculate in 2D.

$$\Gamma^{\theta}{}_{\theta\theta} = \frac{1}{2} g^{\theta\delta} \left(g_{\delta\theta,\theta} + g_{\theta\delta,\theta} - g_{\theta\theta,\delta} \right).$$

Applying the constraints from the metric, this simplifies to

$$\Gamma^{\theta}{}_{\theta\theta} = \frac{1}{2} g^{\theta\theta} \left(g_{\delta\theta,\theta} + g_{\theta\theta,\theta} - g_{\theta\theta,\theta} \right) = 0,$$

since $g_{\theta\theta}$ is constant. Similarly

$$\begin{aligned} \Gamma^{\theta}{}_{\theta\phi} &= \frac{1}{2}g^{\theta\delta}\left(g_{\delta\phi,\theta}+g_{\theta\delta,\phi}-g_{\theta\phi,\delta}\right), \\ &= \frac{1}{2}g^{\theta\theta}\left(g_{\theta\phi,\theta}+g_{\theta\theta,\phi}-g_{\theta\phi,\theta}\right), \\ &= 0, \end{aligned}$$

and thus also $\Gamma^{\theta}{}_{\phi\theta} = 0$.

$$\begin{aligned} \Gamma^{\theta}{}_{\phi\phi} &= \frac{1}{2}g^{\theta\delta}\left(g_{\delta\phi,\phi}+g_{\phi\delta,\phi}-g_{\phi\phi,\delta}\right), \\ &= \frac{1}{2}g^{\theta\theta}\left(g_{\theta\phi,\phi}+g_{\phi\theta,\phi}-g_{\phi\phi,\theta}\right), \\ &= \frac{1}{2}g^{\theta\theta}\left(g_{\theta\phi,\phi}+g_{\phi\theta,\phi}-g_{\phi\phi,\theta}\right), \\ &= \frac{1}{2}R^{-2}\times -\frac{d}{d\theta}(R^2\sin^2\theta), \\ &= -\sin\theta\cos\theta. \end{aligned}$$

$\Gamma^{\phi}{}_{\phi\phi} = 0$ since the metric contains no explicit ϕ dependence.

$$\begin{aligned} \Gamma^{\phi}{}_{\phi\theta} &= \frac{1}{2}g^{\phi\phi}\left(g_{\phi\theta,\phi}+g_{\phi\phi,\theta}-g_{\phi\theta,\phi}\right), \\ &= \frac{1}{2}g^{\phi\phi}\left(g_{\phi\theta,\phi}+g_{\phi\phi,\theta}-g_{\phi\theta,\phi}\right), \\ &= \frac{1}{2}R^{-2}\sin^{-2}\theta\frac{\partial R^2\sin^2\theta}{\partial\theta}, \\ &= \cot\theta. \end{aligned}$$

The last component, $\Gamma^{\phi}{}_{\theta\theta} = 0$.

4.5. (a) Use the Euler-Lagrange equations to show that the shortest path between two points on a sphere satisfies the relations

$$\begin{aligned} \sin^2(\theta)\,\dot{\phi} &= k, \\ \ddot{\theta} &= \dot{\phi}^2\sin\theta\cos\theta, \end{aligned}$$

where k is a constant, and θ and ϕ are the usual angles of spherical polar coordinates.

The Lagrangian can be immediately written as

$$L = R^2\left(\dot{\theta}^2+\sin^2\theta\,\dot{\phi}^2\right).$$

There is no explicit dependence on ϕ and thus $\partial L/\partial\dot{\phi}$ is constant or

$$\sin^2\theta\,\dot{\phi} = \textit{constant},$$

the first of the two relations. For the θ component:

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0,$$

gives

$$\frac{d}{d\lambda}2\dot{\theta} - 2\dot{\phi}^2 \sin\theta\cos\theta = 0,$$

which gives the second relation.

(b) Show that circles of constant longitude satisfy these equations but circles of constant latitude do not, except in one case.

Lines of constant longitude have constant ϕ, therefore $\dot{\phi} = 0$ and the first relation is satisfied, while the second gives $\ddot{\theta} = 0$, which integrates to $\theta = a\lambda + b$ where a and b are constants.

Lines of constant latitude have constant θ, so that $\ddot{\theta} = 0$. This means $\dot{\phi} = 0$, unless $\sin\theta\cos\theta = 0$*. Thus they do not satisfy the relations unless $\theta = 0$ (North pole, and thus not a "path") or $\theta = \pi/2$, the equator.*

4.6. (a) Starting from the flat spacetime metric

$$ds^2 = c^2\,dT^2 - dX^2 - dY^2 - dZ^2,$$

and applying the coordinate transform

$$\begin{aligned} T &= t, \\ X &= x\cos\omega t - y\sin\omega t, \\ Y &= x\sin\omega t + y\cos\omega t, \\ Z &= z, \end{aligned}$$

show that the metric becomes:

$$ds^2 = \left[c^2 - \omega^2\left(x^2+y^2\right)\right]dt^2 + 2\omega y\,dx\,dt - 2\omega x\,dy\,dt - dx^2 - dy^2 - dz^2.$$

Rather than directly applying the tensor transformation

$$g_{\alpha'\beta'} = \frac{\partial x^\gamma}{\partial x^{\alpha'}}\frac{\partial x^\delta}{\partial x^{\beta'}} g_{\gamma\delta},$$

and then writing $ds^2 = g_{\alpha'\beta'}\,dx^{\alpha'}\,dx^{\beta'}$ in practice it is often easier (but completely equivalent) to work out the old differentials dx^α in terms of the new ones, and then work them through the metric. This is an example of this. The only difficulty is taking care over the algebra which can become unpleasant. Here we have

$$\begin{aligned} dT &= dt, \\ dX &= \cos(\omega t)\,dx - \sin(\omega t)\,dy - (\omega x\sin\omega t + \omega y\cos\omega t)\,dt, \\ dY &= \sin(\omega t)\,dx + \cos(\omega t)\,dy + (\omega x\cos\omega t - \omega y\sin\omega t)\,dt, \\ dZ &= dz. \end{aligned}$$

Therefore, substituting (and this is where it gets unpleasant):

$$\begin{aligned} ds^2 &= c^2\,dt^2 - \cos^2\omega t\,dx^2 - \sin^2\omega t\,dy^2 - (\omega x\sin\omega t + \omega y\cos\omega t)^2\,dt^2 + \\ &\quad 2\cos\omega t\sin\omega t\,dx\,dy - 2\sin\omega t(\omega x\sin\omega t + \omega y\cos\omega t)\,dy\,dt + \\ &\quad 2\cos\omega t(\omega x\sin\omega t + \omega y\cos\omega t)\,dx\,dt - \sin^2\omega t\,dx^2 - \cos^2\omega t\,dy^2 - \\ &\quad (\omega x\cos\omega t - \omega y\sin\omega t)^2\,dt^2 - 2\sin\omega t\cos\omega t\,dx\,dy - \\ &\quad 2\sin\omega t(\omega x\cos\omega t - \omega y\sin\omega t)\,dx\,dt - 2\cos\omega t(\omega x\cos\omega t - \omega y\sin\omega t)\,dy\,dt, \\ &\quad -dz^2. \end{aligned}$$

Collecting all terms in dt^2, dx^2 etc leads to the new metric.

(b) Write down the Lagrangian corresponding to this metric.

Replace all differentials by equivalent derivatives:

$$L = \left(c^2 - \omega^2\left(x^2+y^2\right)\right)\dot{t}^2 + 2\omega y\,\dot{x}\,\dot{t} - 2\omega x\,\dot{y}\,\dot{t} - \dot{x}^2 - \dot{y}^2 - \dot{z}^2.$$

(c) Apply the Euler-Lagrange equations to this Lagrangian and show that the equations of geodesic motion are:

$$\begin{aligned} \ddot{t} &= 0, \\ \ddot{x} - \omega^2 x\dot{t}^2 - 2\omega\dot{y}\dot{t} &= 0, \\ \ddot{y} - \omega^2 y\dot{t}^2 + 2\omega\dot{x}\dot{t} &= 0, \\ \ddot{z} &= 0, \end{aligned}$$

where the dots represent differentiation with respect to proper time.

The x-cpt gives

$$\frac{d}{d\lambda}[-2\dot{x} + 2\omega y\dot{t}] + 2\omega^2 x\dot{t}^2 + 2\omega\dot{y}\dot{t} = 0,$$

and thus

$$\ddot{x} - 2\omega\dot{y}\dot{t} - \omega^2 x\dot{t}^2 - \omega y\ddot{t} = 0. \tag{6}$$

Similarly the y-cpt gives

$$\ddot{y} + 2\omega\dot{x}\dot{t} - \omega^2 y\dot{t}^2 + \omega x\ddot{t} = 0. \tag{7}$$

The t-cpt gives

$$\frac{d}{d\lambda}\left[2\left(c^2 - \omega^2\left(x^2+y^2\right)\right)\dot{t} + 2\omega y\dot{x} - 2\omega x\dot{y}\right] = 0,$$

which becomes

$$\left(c^2 - \omega^2\left(x^2+y^2\right)\right)\ddot{t} - 2\omega^2\left(x\dot{x} + y\dot{y}\right)\dot{t} + \omega\left(y\ddot{x} - x\ddot{y}\right) = 0. \tag{8}$$

We can replace the last term in (8) by taking $y\times$ (6) minus $x\times$ (7) which leads to

$$y\ddot{x} - x\ddot{y} - 2\omega(y\dot{y} + x\dot{x})\dot{t} - \omega^2(yx - xy)\dot{t}^2 - \omega(x^2+y^2)\ddot{t} = 0.$$

Substituting this into (8) leaves

$$c^2\ddot{t} = 0.$$

Thus, including the trivial z-cpt, and eliminating the $\ddot{t}$ terms from (6) and (7) we are left with

$$\begin{aligned}
\ddot{t} &= 0,\\
\ddot{x} - \omega^2 x\dot{t}^2 - 2\omega\dot{y}\dot{t} &= 0,\\
\ddot{y} - \omega^2 y\dot{t}^2 + 2\omega\dot{x}\dot{t} &= 0,\\
\ddot{z} &= 0.
\end{aligned}$$

(d) Write down the value of the connection coefficients ${\Gamma^x}_{yt}$ and ${\Gamma^y}_{tt}$ from the above equations.

The general equations of motion can be written as

$$\ddot{x}^\alpha + {\Gamma^\alpha}_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = 0.$$

Therefore ${\Gamma^x}_{yt}$ gives rise to a term in $\dot{y}\dot{t}$ in the x-cpt equation, as does ${\Gamma^x}_{ty}$. Since the connection is symmetric in its lower indices we can write

$${\Gamma^x}_{yt}\dot{y}\dot{t} + {\Gamma^x}_{ty}\dot{t}\dot{y} = 2{\Gamma^x}_{yt}\dot{y}\dot{t},$$

and thus comparing with the x-cpt equation, ${\Gamma^x}_{yt} = -\omega$. Similarly, ${\Gamma^y}_{tt} = -\omega^2 y$.

(e) Introducing the three-vectors $\mathbf{r} = (x, y, z)$ and $\boldsymbol{\omega} = (0, 0, \omega)$, show from these equations that

$$\frac{d^2\mathbf{r}}{dt^2} + \boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r} + 2\boldsymbol{\omega}\times\frac{d\mathbf{r}}{dt} = 0.$$

Since $\dot{t}$ is constant,

$$\frac{d}{d\lambda} = \dot{t}\frac{d}{dt},$$

and

$$\frac{d^2}{d\lambda^2} = \frac{d}{d\lambda}\dot{t}\frac{d}{dt} = \dot{t}^2\frac{d^2}{dt^2},$$

and so the spatial component equations can be written

$$\begin{aligned}
\ddot{x} - \omega^2 x - 2\omega\dot{y} &= 0,\\
\ddot{y} - \omega^2 y + 2\omega\dot{x} &= 0,\\
\ddot{z} &= 0,
\end{aligned}$$

where the dots now represent derivative swith respect to t rather than an affine parameter. $\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r}$ *can be written*

$$\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{r} = (\boldsymbol{\omega}\cdot\mathbf{r})\boldsymbol{\omega} - \omega^2\mathbf{r},$$

and, since $\boldsymbol{\omega} = \omega\hat{\vec{z}}$, then

$$(\boldsymbol{\omega}\cdot\mathbf{r})\boldsymbol{\omega} = (\omega^2 z)\,\hat{\vec{z}},$$

and so $\boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r}$ *reduces to* $(-\omega^2 x, -\omega^2 y, 0)$. *Similarly*

$$\boldsymbol{\omega} \times \dot{\mathbf{r}} = (-\omega \dot{y}, \omega \dot{x}, 0),$$

and the relation is proved.

What are the physical meanings of the terms in this expression?

The three terms are (i) acceleration in terms of the new coordinates, (ii) centripetal, (iii) Coriolis.

Why have they arisen from the coordinate transformation specified?

The new coordinates are fixed in a rotating frame, thus the appearance of "fictitious" accelerations due to the use of a non-inertial frame of reference. Note that the simple appearance of the equations is mis-leading since the derivatives are in terms of coordinate, not proper time. Note too that g_{tt} *switches sign when* $\omega^2(x^2 + y^2) > c^2$. *Time-like world-lines then require that the spatial coordinates cannot remain fixed. Something similar occurs at the Schwarzschild radius of black-holes. In the case here it can be seen simply as a consequence of nothing travelling faster than light.*

4.7. Show that the covariant derivative obeys Leibniz' product rule, e.g. given a tensor $T^{\alpha\beta} = U^\alpha V^\beta$, show that

$$T^{\alpha\beta}{}_{;\gamma} = U^\alpha{}_{;\gamma} V^\beta + U^\alpha V^\beta{}_{;\gamma}.$$

The left-hand side can be written

$$\begin{aligned} T^{\alpha\beta}{}_{;\gamma} &= T^{\alpha\beta}{}_{,\gamma} + \Gamma^\alpha{}_{\sigma\gamma} T^{\sigma\beta} + \Gamma^\beta{}_{\sigma\gamma} T^{\alpha\sigma}, \\ &= \left(U^\alpha V^\beta\right)_{,\gamma} + \Gamma^\alpha{}_{\sigma\gamma} U^\sigma V^\beta + \Gamma^\beta{}_{\sigma\gamma} U^\alpha V^\sigma, \\ &= U^\alpha{}_{,\gamma} V^\beta + U^\alpha V^\beta{}_{,\gamma} + \Gamma^\alpha{}_{\sigma\gamma} U^\sigma V^\beta + \Gamma^\beta{}_{\sigma\gamma} U^\alpha V^\sigma, \\ &= \left[U^\alpha{}_{,\gamma} + \Gamma^\alpha{}_{\sigma\gamma} U^\sigma\right] V^\beta + U^\alpha \left[V^\beta{}_{,\gamma} + \Gamma^\beta{}_{\sigma\gamma} V^\sigma\right], \\ &= U^\alpha{}_{;\gamma} V^\beta + U^\alpha V^\beta{}_{;\gamma}, \end{aligned}$$

QED.

4.8. The interval of a static, spherically symmetric spacetime can be written

$$ds^2 = A(r)\,dt^2 - B(r)\,dr^2 - r^2\,d\theta^2 - r^2 \sin^2\theta\,d\phi^2.$$

(a) Use the Euler-Lagrange equations to derive the equations of geodesic motion and hence show that $\Gamma^t{}_{tr} = A'/2A$, $\Gamma^r{}_{tt} = A'/2B$, $\Gamma^r{}_{rr} = B'/2B$, $\Gamma^r{}_{\theta\theta} = -r/B$, $\Gamma^r{}_{\phi\phi} = -(r\sin^2\theta)/B$, $\Gamma^\theta{}_{r\theta} = 1/r$, $\Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta$, $\Gamma^\phi{}_{r\phi} = 1/r$, $\Gamma^\phi{}_{\theta\phi} = \cot\theta$, with all others zero. The dashes here denote differentiation with respect to r

[These coefficients are needed for Schwarzschild's solution. You need not calculate them all, just make sure that you understand how to go about it in principle.]

The equivalent Lagrangian is

$$L = A(r)\,\dot{t}^2 - B(r)\,\dot{r}^2 - r^2\,\dot{\theta}^2 - r^2 \sin^2\theta\,\dot{\phi}^2.$$

Consider the r-component, then:

$$\frac{d}{d\lambda}\left[-2B\dot{r}\right] - \left[A'\dot{t}^2 - B'\dot{r}^2 - 2r\dot{\theta}^2 - 2r\sin^2\theta\dot{\phi}^2\right] = 0,$$

so, applying the product rule to the first term and using $dB/d\lambda = (dB/dr)(dr/d\lambda) = B'\dot{r}$,

$$-2B'\dot{r}^2 - 2B\ddot{r} - \left[A'\dot{t}^2 - B'\dot{r}^2 - 2r\dot{\theta}^2 - 2r\sin^2\theta\dot{\phi}^2\right] = 0,$$

and therefore collecting terms,

$$\ddot{r} + \frac{1}{2B}\left[A'\dot{t}^2 + B'\dot{r}^2 - 2r\dot{\theta}^2 - 2r\sin^2\theta\dot{\phi}^2\right] = 0.$$

It is then straight-forward to read off the connection coefficients with r as the upper index, e.g. ${\Gamma^r}_{tt} = A'/2B$, ${\Gamma^r}_{rr} = B'/2B$, because the general equations of motion are $\ddot{x}^\alpha + {\Gamma^\alpha}_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = 0$.

(b) The "proper" acceleration, a, is the acceleration felt by an observer and can be calculated from the norm of the four-acceleration:

$$a^2 = -\vec{A}\cdot\vec{A} = -g_{\alpha\beta}A^\alpha A^\beta.$$

Use this to show that the proper acceleration felt by an astronaut who uses a rocket to remain stationary in terms of the coordinates (r, θ, ϕ) is given by

$$a^2 = \frac{1}{B}\left(\frac{c^2 A'}{2A}\right)^2.$$

Constant position $\Longrightarrow$ all spatial components fixed, e.g. $\dot{r} = \dot{\theta} = \dot{\phi} = 0$, and so $\vec{U} = (\dot{t}, 0, 0, 0)$. Since $\vec{A}\cdot\vec{U} = 0$, then $A^t = 0$, leaving only spatial components

$$A^i = \ddot{x}^i + {\Gamma^i}_{\beta\gamma}\dot{x}^\alpha\dot{x}^\beta,$$

for $i = 1$, 2 or 3. The constant position ($\dot{x}^i = \ddot{x}^i = 0$) reduces this to

$$A^i = {\Gamma^i}_{tt}\dot{t}^2.$$

The only non-zero connection coefficient of type ${\Gamma^i}_{tt}$ is

$${\Gamma^r}_{tt} = \frac{A'}{2B},$$

so $A^r = \Gamma^r{}_{tt}\dot{t}^2$ *is the only component of acceleration, and* $a^2 = -g_{rr}(A^r)^2$*. With* $dr = d\theta = d\phi = 0$ *we have*

$$ds^2 = c^2\,d\tau^2 = A\,dt^2,$$

so

$$\dot{t}^2 = \frac{c^2}{A},$$

and since $g_{rr} = -B$*, we finally obtain*

$$a^2 = -g_{rr}\left(\Gamma^r{}_{tt}\dot{t}^2\right)^2 = B\left(\frac{c^2 A'}{2AB}\right)^2,$$

which gives the answer of the question. For the Schwarzschild metric this reduces to

$$a = \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2}\frac{GM}{r^2},$$

the same as Newton at large r*, but* $\to \infty$ *as* $r \to 2GM/c^2$*.*

(c) What is the proper acceleration of an astronaut following a geodesic path in this metric?

Zero: geodesic motion is the same as free-fall, in which one feels no acceleration.

4.9. * It is possible to have a connection in a space without a metric; in this sense it is a more fundamental concept. For instance, independently of any metric one can derive the transformation properties of the connection in order that the following

$$V^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\gamma\beta}V^\gamma,$$

are the components of a tensor.

(a) Use this approach to show that the connection must transform as follows

$$\Gamma^{\alpha'}{}_{\beta'\gamma'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial x^\gamma}{\partial x^{\gamma'}}\Gamma^\alpha{}_{\beta\gamma} - \frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial x^\gamma}{\partial x^{\gamma'}}\frac{\partial^2 x^{\alpha'}}{\partial x^\beta \partial x^\gamma}.$$

By definition, if these are tensor components then

$$V^{\alpha'}{}_{,\beta'} + \Gamma^{\alpha'}{}_{\gamma'\beta'}V^{\gamma'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x^{\beta'}}\left(V^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\gamma\beta}V^\gamma\right).$$

Replacing components in the dashed frame by components in the undashed frame, the left-hand side can be written as

$$\frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial}{\partial x^\beta}\left(\frac{\partial x^{\alpha'}}{\partial x^\alpha}V^\alpha\right) + \Gamma^{\alpha'}{}_{\gamma'\beta'}\frac{\partial x^{\gamma'}}{\partial x^\gamma}V^\gamma,$$

and thus

$$\frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial^2 x^{\alpha'}}{\partial x^\beta \partial x^\alpha}V^\alpha + \frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial x^{\alpha'}}{\partial x^\alpha}V^\alpha{}_{,\beta} + \Gamma^{\alpha'}{}_{\gamma'\beta'}\frac{\partial x^{\gamma'}}{\partial x^\gamma}V^\gamma.$$

The second term in this equation cancels with the first term on the right of the first equation and we are left with

$$\frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial^2 x^{\alpha'}}{\partial x^\beta \partial x^\alpha}V^\alpha + \Gamma^{\alpha'}{}_{\gamma'\beta'}\frac{\partial x^{\gamma'}}{\partial x^\gamma}V^\gamma = \frac{\partial x^{\alpha'}}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x^{\beta'}}\Gamma^\alpha{}_{\gamma\beta}V^\gamma.$$

Re-labelling α to γ in the first term and remembering that $\vec{V}$ is arbitrary, we can immediately write

$$\frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial^2 x^{\alpha'}}{\partial x^\beta \partial x^\gamma} + \Gamma^{\alpha'}{}_{\gamma'\beta'}\frac{\partial x^{\gamma'}}{\partial x^\gamma} = \frac{\partial x^{\alpha'}}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x^{\beta'}}\Gamma^\alpha{}_{\gamma\beta}.$$

Multiplying by $\partial x^\gamma/\partial x^{\delta'}$ and contracting γ and using

$$\frac{\partial x^\gamma}{\partial x^{\delta'}}\frac{\partial x^{\gamma'}}{\partial x^\gamma} = \delta^{\gamma'}_{\delta'},$$

leads to

$$\frac{\partial x^\gamma}{\partial x^{\delta'}}\frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial^2 x^{\alpha'}}{\partial x^\beta \partial x^\gamma} + \Gamma^{\alpha'}{}_{\delta'\beta'} = \frac{\partial x^\gamma}{\partial x^{\delta'}}\frac{\partial x^{\alpha'}}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x^{\beta'}}\Gamma^\alpha{}_{\gamma\beta}.$$

Finally re-labelling δ' to β', β' to γ', γ to β and β to γ and re-arranging gives

$$\Gamma^{\alpha'}{}_{\beta'\gamma'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial x^\gamma}{\partial x^{\gamma'}}\Gamma^\alpha{}_{\beta\gamma} - \frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial x^\gamma}{\partial x^{\gamma'}}\frac{\partial^2 x^{\alpha'}}{\partial x^\beta \partial x^\gamma},$$

QED.

(b) Show therefore that

$$T^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\beta\gamma} - \Gamma^\alpha{}_{\gamma\beta},$$

is a tensor, known as the *torsion*. In GR the connection is assumed to be torsionless and therefore symmetric in its lower indices.

This can be shown starting from

$$T^{\alpha'}{}_{\beta'\gamma'} = \Gamma^{\alpha'}{}_{\beta'\gamma'} - \Gamma^{\alpha'}{}_{\gamma'\beta'}.$$

Replacing the connections using the transformation relation, the second derivative terms cancel because

$$\frac{\partial^2 x^{\alpha'}}{\partial x^\beta \partial x^\gamma} = \frac{\partial^2 x^{\alpha'}}{\partial x^\gamma \partial x^\beta},$$

leaving

$$T^{\alpha'}{}_{\beta'\gamma'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial x^\gamma}{\partial x^{\gamma'}}T^\alpha{}_{\beta\gamma},$$

the transformation relation of a tensor.

4.10. * The connection is not unique to GR and can come up in Newtonian physics applied to curved coordinate systems, although often more elementary means can lead to the same results. Consider for example the continuity equation of fluid mechanics:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,$$

where ρ is the density and $\mathbf{v}$ is the fluid velocity. By expressing the second term covariantly (i.e. in a way that is tensorially correct in curved coordinate systems which reduces to the usual expression in Cartesian coordinates) show that in cylindrical polar coordinates this is given by

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0,$$

where (v_r, v_θ, v_z) are the components of velocity in the usual orthogonal unit basis vectors aligned with the polar coordinates.

The continuity equation can be written covariantly using the connection as

$$\frac{\partial \rho}{\partial t} + \partial_i(\rho v^i) + \Gamma^i{}_{ki}(\rho v^k) = 0,$$

where i and k denote the r, θ and z components. As discussed in an earlier problem on coordinate bases, the components of the velocity vector (since I wrote v^i above, not v_i) in a basis defined by (r, θ, z) are $(v_r, v_\theta/r, v_z)$. The only non-zero values of the connection are $\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = 1/r$ and $\Gamma^r{}_{\theta\theta} = -r$, and of these only $\Gamma^\theta{}_{r\theta} = 1/r$ is relevant given the term $\Gamma^i{}_{ki}$ in the covariant divergence. The continuity equation can thus be written as

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}(\rho v_r) + \frac{\partial}{\partial \theta}(\rho v_\theta/r) + \frac{\partial}{\partial z}(\rho v_z) + \frac{\rho v_r}{r} = 0.$$

The second and last terms can be combined into one while the r can be moved outside the θ derivative to arrive at the expression given.

The arrangement of indices at the start is not unique. For instance one could instead have started from

$$\frac{\partial \rho}{\partial t} + g^{ij}\partial_j(\rho v_i) - g^{ij}\Gamma^k{}_{ij}(\rho v_k) = 0,$$

taking care to balance indices and use the appropriate sign on the connection. Raising and lowering indices, this can be re-written as

$$\frac{\partial \rho}{\partial t} + g^{ij}\partial_j(\rho g_{ik} v^k) - g^{ij}\Gamma^k{}_{ij}(\rho g_{km} v^m) = 0.$$

The metric coefficients are given by $g_{rr} = g^{rr} = 1$, $g_{\theta\theta} = r^2 = 1/g^{\theta\theta}$ and $g_{zz} = g^{zz} = 1$. The only relevant connection coefficient is $\Gamma^r{}_{\theta\theta} = -r$ and thus one obtains

$$\frac{\partial \rho}{\partial t} + \partial_r(\rho v_r) + r^{-2}\partial_\theta(\rho r^2 v_\theta/r) + \partial_z(\rho v_z) + r^{-2}(r\rho v_r) = 0,$$

leading to the same equation as before. The arrangement of indices is not unique because there is no distinction between co- and contra-variant indices in Cartesian coordinates. Clearly however the first approach was more direct.

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[1] Hence show that in an axi-symmetric accretion disc in which v_r is independent of z

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\Sigma v_r) = 0,$$

where Σ is the vertically-integrated surface density of the disc.

Axi-symmetry implies no dependence upon θ leaving

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{\partial}{\partial z}(\rho v_z) = 0.$$

Integrating over all z:

$$\int_{-\infty}^{\infty} \frac{\partial \rho}{\partial t}\, dz + \int_{-\infty}^{\infty} \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r)\, dz + \int_{-\infty}^{\infty} \frac{\partial}{\partial z}(\rho v_z)\, dz = 0.$$

The order of the derivatives and integrals can be swapped on the first two terms, while the third gives

$$\int_{-\infty}^{\infty} \frac{\partial}{\partial z}(\rho v_z)\, dz = [\rho v_z]_{-\infty}^{\infty} = 0,$$

since the density of the disc can be assumed to drop to zero far from the disc. The integration over z can be applied to ρ alone since both r and v_r are independent of z. Since

$$\Sigma = \int_{-\infty}^{\infty} \rho\, dz,$$

we are left with the expression as given.

4.11. * There is no need to distinguish between "up" and "down" indices in Cartesian coordinates and conventionally all indices are down. Thus the Laplacian operator can be written as $\nabla^2 \psi = \partial_i \partial_i \psi$ in Cartesian tensor notation.

Work out a fully-covariant form of the Laplacian and hence prove the well-known-but-rarely-proven relation for the Laplacian in spherical polars:

$$\nabla^2 \psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2}.$$

The following is manifestly covariant (covariant derivative of the gradient <u>vector</u> contracted on its two indices):

$$\nabla^2 \psi = \partial_i \left(\partial^i \psi\right) + \Gamma^i{}_{ki} \left(\partial^k \psi\right),$$

and reduces to the Cartesian form of the Laplacian in Cartesian coordinates in Euclidean 3D since the connection then disappears. The Levi-Civita connection

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{im} \left(g_{mk,j} + g_{jm,k} - g_{jk,m}\right),$$

[1] This part is very off-topic so don't worry if you can't manage it.

with j set to k and k to i becomes

$$\Gamma^{i}{}_{ki} = \frac{1}{2} g^{im} \left(g_{mi,k} + g_{km,i} - g_{ki,m} \right).$$

However since both i and m are dummy indices and since g is symmetric, the last two terms cancel leaving

$$\Gamma^{i}{}_{ki} = \frac{1}{2} g^{im} g_{mi,k}.$$

Thus, lowering the indices on the derivative terms, the Laplacian becomes

$$\nabla^2 \psi = \partial_i \left(g^{ij} \partial_i \psi \right) + \frac{1}{2} g^{im} g_{mi,k} g^{kn} \partial_n \psi.$$

There are simpler forms of this, but this will do for our purposes here.

The metric in spherical polars is $g_{rr} = g^{rr} = 1$, $g_{\theta\theta} = 1/g^{\theta\theta} = r^2$, $g_{\phi\phi} = 1/g^{\phi\phi} = r^2 \sin^2\theta$, all other terms being zero. Therefore, keeping a cool head, one can expand the various terms to obtain

$$\begin{aligned}
\nabla^2 \psi &= \partial_r^2 \psi + \partial_\theta \left(r^{-2} \partial_\theta \psi \right) + \partial_\phi \left(r^{-2} \sin^{-2}\theta \partial_\phi \psi \right) + \\
&\quad \frac{1}{2} \left(r^{-2} \times 2r \, \partial_r \psi + r^{-2} \sin^{-2}\theta \times 2r \sin^2\theta \, \partial_r \psi + r^{-2} \sin^{-2}\theta \times 2r^2 \sin\theta \cos\theta \times r^{-2} \, \partial_\theta \psi \right), \\
&= \frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\cos\theta}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta} + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \psi}{\partial \phi^2}.
\end{aligned}$$

The first two pairs of terms can be combined as follows:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) = \frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r} \frac{\partial \psi}{\partial r},$$

and

$$\frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \psi}{\partial \theta} \right) = \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\cos\theta}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta}.$$

The standard form of the Laplacian in spherical polars follows directly. Given the effort needed to develop the mathematical machinery of tensors in curved coordinates systems and to solve this problem, you can probably appreciate why this derivation is not usually covered along the way to solving for the eigenstates of the hydrogen atom for instance.