Department of Physics
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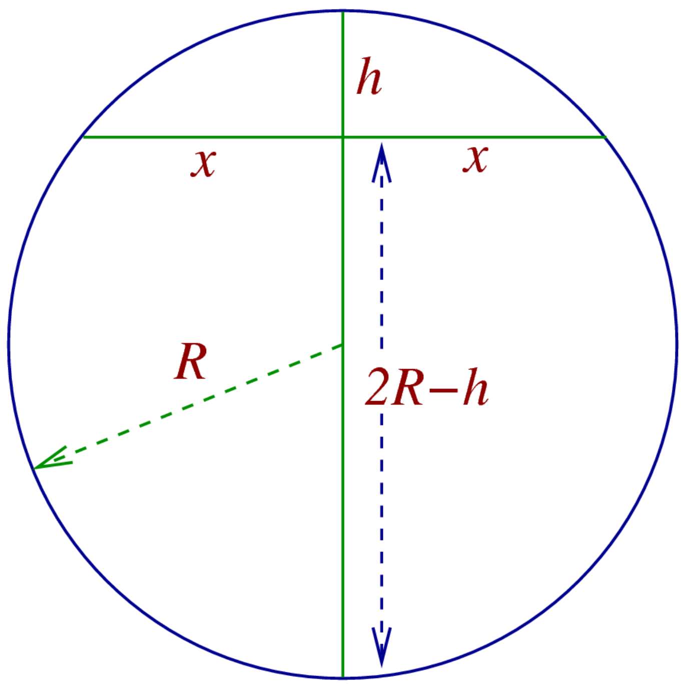
PX436, General Relativity Problems *with answers*

1.1. Calculate the radius at which the bending of light as calculated from the equivalence principle would keep it in a circular orbit around a mass M.

From lectures, over a distance x in a gravitational field g, light is deflected down by an amount

$$h = \frac{gx^2}{2c^2}.$$

Consider the following:



A basic theorem of the geometry of circles gives

$$x^2 = (2R - h)h \approx 2Rh,$$

for small h. Therefore

$$h = \frac{g(2Rh)}{2c^2},$$

so

$$gR = \frac{GM}{R} = c^2,$$

or

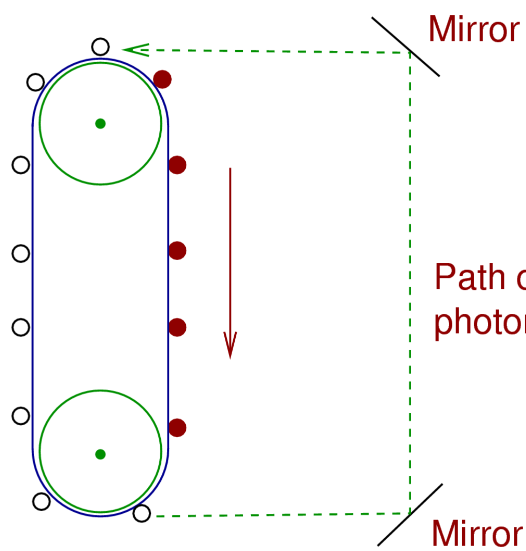
$$R = \frac{GM}{c^2}.$$

Alternatively, and more easily, but not immediately so obviously, balance centripetal acceleration and gravity

$$\frac{v^2}{R} = \frac{c^2}{R} = \frac{GM}{R^2},$$

which leads to the same result, exactly as Newton would have deduced. This is a nice example where the Newtonian calculation fails and we shall see later in the course that the correct answer is $R = 3GM/c^2$. The equivalence principle alone as applied here does not give the right answer either; full GR is required to extend the calculation of light-bending over large regions in non-uniform fields.

1.2. The diagram below shows the design of a perpetual motion machine sent by an inventor called H.Bondi to a patent office in Switzerland where a certain A.Einstein works:



Atoms attached to a movable belt absorb photons at the top of their travel and emit them at the bottom, with the photons directed back to the top-most atoms where they are absorbed. When an atom absorbs a photon of energy $E = h\nu$ its mass increases by $\Delta m = h\nu/c^2$ hence the excited atoms (filled circles) on the right of the belt always outweigh the de-excited atoms (empty circles) on the left and hence perpetual clockwise motion results which can solve all the world's energy problems ...

What flaw does Einstein spot in this machine in addition to the evident violation of the first law of thermodynamics?

There is an implicit assumption that the photon can be re-cycled with no change in energy, but of course it loses energy as it climbs the gravitational potential as shown from the equivalence principle in lectures. In fact, the change in frequency is given by $\Delta\nu = -\nu_0 \Delta\phi/c^2$, where $\Delta\phi$ is *the change of gravitational potential that the photon undergoes. This corresponds to a change in energy of*

$$\Delta E = -\frac{h\nu_0}{c^2}\Delta\phi = -\Delta m \Delta\phi,$$

compensating for the energy extracted from the heavier atoms.

1.3. Why can there be no equivalent of Einstein's equivalence principle for electric rather than gravitational fields?

Because the ratio of electric charge to inertial mass is variable. Thus not all objects have the same acceleration in an electric field and there is no unique "free-fall" frame.

1.4. It is important to be confident with the summation convention and index manipulation. Here are a few practice problems:

(a) Show that

$$A^\alpha B_\alpha = A^\beta B_\beta.$$

Writing out the left side

$$A^\alpha B_\alpha = A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3.$$

Since the right-hand side is independent of the dummy index α, the relation is obvious.

(b) What is the value of δ^α_α?

$\delta^0_0 + \delta^1_1 + \delta^2_2 + \delta^3_3 = 4.$

(c) Show that in SR

$$\vec{A}\cdot\vec{A} = \left(A^0\right)^2 - A^i A^i,$$

where $\vec{A}$ is an arbitrary four vector and in this case the summation convention applies to the last term even though both indices are raised.

By definition

$$\vec{A}\cdot\vec{A} = \eta_{\alpha\beta}A^\alpha A^\beta.$$

Substituting the values of the SR metric, the relation follows.

(d) If $A_{\alpha\beta}$ is symmetric ($A_{\alpha\beta} = A_{\beta\alpha}$), and $B^{\alpha\beta}$ is anti-symmetric ($B^{\alpha\beta} = -B^{\beta\alpha}$), show that $A_{\alpha\beta}B^{\alpha\beta} = 0$.

From the relations

$$A_{\alpha\beta}B^{\alpha\beta} = (A_{\beta\alpha})\left(-B^{\beta\alpha}\right).$$

But α and β are dummy indices which are summed over and we can simply re-label so that they are swapped giving

$$A_{\alpha\beta}B^{\alpha\beta} = -A_{\alpha\beta}B^{\alpha\beta}.$$

This implies that

$$A_{\alpha\beta}B^{\alpha\beta} = 0,$$

QED.

(e) Show that, if $A_{\alpha\beta}$ is symmetric but $B^{\alpha\beta}$ is arbitrary, then

$$A_{\alpha\beta}B^{\alpha\beta} = A_{\alpha\beta}C^{\alpha\beta},$$

where

$$C^{\alpha\beta} = \frac{1}{2}\left(B^{\alpha\beta} + B^{\beta\alpha}\right).$$

Proceeding as with the previous part but without initially swapping the indices of $B_{\alpha\beta}$ one can show

$$A_{\alpha\beta}B^{\alpha\beta} = A_{\alpha\beta}B^{\beta\alpha}.$$

Adding these two and dividing by 2 gives the expression asked for.

(f) How many independent components are needed to specify $A^{\alpha\beta}$ when it is (i) arbitrary, (ii) symmetric and (iii) anti-symmetric in 3, 4 and N dimensions?

(i) 9, 16, N^2. (ii) Symmetry implies $A^{\alpha\beta} = A^{\beta\alpha}$. There are $N(N-1)/$ such conditions, so the total number of components $= N^2 - N(N-1)/2 = N(N+1)/2$, which gives 6 and 10 components for 3 and 4 dimensions. (iii) Anti-symmetry implies the same $N(N-1)/2$ conditions plus N conditions of the form $A^{00} = 0$. This gives $N(N-1)/2$ components giving 3 *and* 6 *components respectively.*

(g) The "Christoffel symbols", $\Gamma^{\alpha}{}_{\rho\sigma}$, are symmetric in the indices ρ and σ. How many independent components are there in four dimensions?

From the previous part, there are 10 different combinations of ρ and σ for any one value of α. Since there are four values of α, there are 40 independent Christoffel symbols.

1.5. If t and x transform according to

$$\begin{aligned} ct' &= \alpha(ct) + \beta x, \\ x' &= \gamma(ct) + \delta x, \end{aligned}$$

where α, β, γ and δ are constants, such that the interval $s^2 = (ct)^2 - x^2$ is preserved, show that $\alpha^2 - \beta^2 = 1$, $\gamma = \beta$ and $\delta = \alpha$.

$$(ct')^2 - (x')^2 = \left(\alpha^2 - \gamma^2\right)(ct)^2 + 2(\alpha\beta - \gamma\delta)(ct)x - (\delta^2 - \beta^2)x^2.$$

Therefore

$$\begin{aligned} \alpha^2 - \gamma^2 &= 1, \\ \alpha\beta &= \gamma\delta, \\ \delta^2 - \beta^2 &= 1. \end{aligned}$$

Squaring the second of these equations and using the other two

$$\alpha^2\beta^2 = \left(\alpha^2 - 1\right)\left(\beta^2 + 1\right),$$

thus

$$\alpha^2 - \beta^2 = 1.$$

Since $\alpha^2 - \gamma^2 = 1$, *then* $\gamma = \pm\beta$, *and from* $\alpha\beta = \gamma\delta$ *then* $\delta = \pm\alpha$, *with the sign being the same in each relation. For continuity, we take the positive sign so* $\delta = \alpha$, $\gamma = \beta$, *and the relations are proved.*

Show that these relations are consistent with standard LTs.

For the standard LT, set $\alpha \to \gamma$, *and* $\beta \to -\gamma\beta$, *where* $\gamma = 1/\sqrt{1-\beta^2}$ *(where* γ *and* β *are not to be confused with the constants of the linear transform). Thus*

$$\alpha^2 - \beta^2 = \frac{1}{1-\beta^2} - \frac{\beta^2}{1-\beta^2} = 1.$$

The other relations are obviously satisfied.

1.6. Use the component transformation definition of vectors to show that if $\vec{A}$ and $\vec{B}$ are four-vectors then the quantities V^α defined by $V^\alpha = A^\alpha + B^\alpha$ also transform as a four-vector enabling one to write $\vec{V} = \vec{A} + \vec{B}$.

Since $\vec{A}$ *and* $\vec{B}$ *are four-vectors then we can write*

$$\begin{aligned} A^{\alpha'} &= \Lambda^{\alpha'}{}_{\beta} A^\beta, \\ B^{\alpha'} &= \Lambda^{\alpha'}{}_{\beta} B^\beta. \end{aligned}$$

Therefore

$$\begin{aligned}\Lambda^{\alpha'}{}_{\beta} V^{\beta} &= \Lambda^{\alpha'}{}_{\beta}\left(A^{\beta}+B^{\beta}\right),\\ &= \Lambda^{\alpha'}{}_{\beta} A^{\beta}+\Lambda^{\alpha'}{}_{\beta} B^{\beta},\\ &= A^{\alpha'}+B^{\alpha'}.\end{aligned}$$

QED.

1.7. The four-velocity in Special Relativity (SR), $\vec{U}$, has components

$$U^{\alpha}=\frac{dx^{\alpha}}{d\tau}=\gamma(c,\vec{v}),$$

where in the first representation x^{α} represent the coordinates of events in the usual representation with $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$, and τ is the proper time, while in the second, γ is the Lorentz factor and $\vec{v}$ the three-velocity.

(a) Why, given the properties of dx^{α} and τ, is $\vec{U}$ "clearly" a four-vector?

Since $\vec{x}$ is a four-vector then $d\vec{x}$ is also a four-vector. Since τ is a scalar, then so too is $d\tau$, and therefore $\vec{U} = d\vec{x}/d\tau$ is a four-vector.

(b) Show that in *any* reference frame

$$\vec{U}\cdot\vec{U}=\eta_{\alpha\beta}U^{\alpha}U^{\beta}=c^2,$$

where $\eta_{\alpha\beta}$ is the SR metric as defined in lectures.

$\vec{U}\cdot\vec{U}$ is a frame-invariant scalar, so we need only work out its value in one frame. Choosing the rest frame $\vec{U} = (c, 0, 0, 0)$, the answer follows directly.

(c) Hence show that

$$\vec{U}\cdot\vec{A}=0,$$

in all frames, where the four-acceleration $\vec{A}$ is defined by

$$A^{\alpha}=\frac{dU^{\alpha}}{d\tau}.$$

This is just a matter of differentiating with respect to proper time τ (not t because that would not be frame-invariant):

$$\begin{aligned}\frac{d}{d\tau}\vec{U}\cdot\vec{U} &= \frac{d}{d\tau}\eta_{\alpha\beta}U^{\alpha}U^{\beta},\\ &= \eta_{\alpha\beta}\frac{dU^{\alpha}}{d\tau}U^{\beta}+\eta_{\alpha\beta}U^{\alpha}\frac{dU^{\beta}}{d\tau},\\ &= 2\eta_{\alpha\beta}U^{\alpha}\frac{dU^{\beta}}{d\tau},\\ &= 2\vec{U}\cdot\vec{A},\end{aligned}$$

where $\vec{A} = d\vec{U}/d\tau$. In the above set of equation line 2 follows from line 1 since the $\eta_{\alpha\beta}$ are constant in SR; line 3 follows from line 2 after swapping α and β and using $\eta_{\alpha\beta} = \eta_{\beta\alpha}$. Since $\vec{U} \cdot \vec{U} = c^2$, its derivative $= 0$ and the result follows.

(d) Show that $\vec{A}$ is given by

$$\vec{A} = \gamma(\dot{\gamma}c, \dot{\gamma}\vec{v} + \gamma\vec{a}),$$

where $\vec{a}$ is the 3-acceleration and the dots indicate differentiation with respect to t.

We can write $\vec{U} = \gamma(c, \vec{v})$, and since $d\tau = dt/\gamma$, we have

$$\vec{A} = \gamma\frac{d\vec{U}}{dt}.$$

Since

$$\begin{aligned}\frac{d\gamma\vec{v}}{dt} &= \dot{\gamma}\vec{v} + \gamma\frac{d\vec{v}}{dt}, \\ &= \dot{\gamma}\vec{v} + \gamma\vec{a},\end{aligned}$$

the result follows straight-forwardly.

(e) An object undergoes acceleration a in the x-direction as measured in its instantaneous rest frame (IRF). Show that in a frame in which it travels with speed v in the x-direction, it has an acceleration

$$a' = \gamma^{-3}a,$$

where $\gamma = \gamma(v)$ is the Lorentz factor.

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*From part (**1.7**d) we can write $\vec{A} = (A^0, \vec{a})$ for the IRF, where also $\vec{U} = (c, \vec{0})$. Since $\vec{U} \cdot \vec{A} = 0$, we must further have $A^0 = 0$, so $\vec{A} = (0, \vec{a}) = (0, a, 0, 0)$ in this case. Applying Lorentz transforms and remembering the general form of $\vec{A}$ from part (**1.7**d) we have for the time component:*

$$\gamma\dot{\gamma}c = \gamma(0 + \frac{v}{c}a),$$

so $\dot{\gamma} = va/c^2$ (remembering that we are transforming from a frame travelling at speed v relative to the frame we are transforming to, hence the + sign on the second term).

Similarly, for the x component,

$$\gamma\dot{\gamma}v + \gamma^2 a' = \gamma\left(a + \frac{v}{c} \times 0\right),$$

so using $\dot{\gamma} = va/c^2$,

$$\gamma\frac{v^2}{c^2}a + \gamma^2 a' = \gamma a,$$

and remembering that $1 - v^2/c^2 = \gamma^{-2}$, the result follows.

(f) Starting from rest with $v = t = \tau = 0$ and $x = c^2/a$, an object undergoes constant acceleration a in the x-direction in its IRF. Integrate the result of part (**1.7**e) to show that:

$$\begin{aligned} v &= c \tanh \frac{a\tau}{c}, \\ \gamma &= \cosh \frac{a\tau}{c}, \\ x &= \frac{c^2}{a} \cosh \frac{a\tau}{c}, \\ t &= \frac{c}{a} \sinh \frac{a\tau}{c}, \end{aligned}$$

where τ is the proper time, i.e the time measured by clocks travelling with the object.

The equation $a' = \gamma^{-3} a$ can be written as

$$\frac{dv}{dt} = \frac{a}{\gamma^3},$$

but as suggested by the form of the answers, it is more easily integrated in terms of proper time for which since $dt = \gamma\, d\tau$,

$$\frac{dv}{d\tau} = \frac{a}{\gamma^2},$$

and so

$$\int_0^v \frac{dv}{1 - v^2/c^2} = \int_0^\tau a\, d\tau.$$

Setting $v = c \tanh z$, then $1 - v^2/c^2 = \cosh^{-2} z$, while $dv = c \cosh^{-2}(z)\, dz$, so we are left with

$$cz = a\tau,$$

and therefore

$$v = c \tanh \frac{a\tau}{c}.$$

The Lorentz factor $\gamma = (1 - v^2/c^2)^{-1/2} = \cosh a\tau/c$ follows immediately. Then

$$\int dt = t = \int \gamma\, d\tau = \frac{c}{a} \sinh \frac{a\tau}{c},$$

follows straightforwardly. Finally the distance is obtained from integration of

$$\frac{dx}{d\tau} = \gamma \frac{dx}{dt} = \gamma v,$$

or

$$\int_{c^2/a}^x dx = \int_0^\tau c \sinh \frac{a\tau}{c}\, d\tau,$$

which gives

$$x - \frac{c^2}{a} = \frac{c^2}{a} \cosh \frac{a\tau}{c} - \frac{c^2}{a},$$

and so the result for x is proven as well.

(g) From the previous part, obtain a single equation relating x and t and hence draw the worldline of the object in a spacetime diagram.

Using $\cosh^2 x - \sinh^2 x = 1$ *gives*

$$x^2 - (ct)^2 = \left(\frac{c^2}{a}\right)^2.$$

This gives a hyperbola, which crosses the x*-axis vertically and asymptotes to the line* $x = ct$.

In what sense does the line $x = ct$ act as a "horizon" for the object?

Any event above this line, i.e. with $ct > x$ *can never be seen by the object because the object's worldline will never cross into its "future". The line* $x = ct$ *is an "event horizon" beyond which the object can* <u>*never*</u> *see.*

(h) * Use the result of the previous part to calculate the shortest time it would take a spacecraft to travel to the centre of our Galaxy 25,000 light-years from Earth and back again assuming that it could maintain a constant IRF acceleration ("proper" acceleration) of one g at all times and that it comes to rest, however briefly, at the Galactic centre. Calculate the time as reckoned by an observer on Earth and on the spacecraft.

Shifting the origin of x *so that it starts at* $x = 0$ *gives*

$$x = \frac{c^2}{a}\left(\cosh\frac{a\tau}{c} - 1\right).$$

Setting $a = g$ *and* $x = cT$ *where* T *is the distance in terms of time, we have*

$$\cosh\frac{g\tau}{c} = 1 + \frac{gT}{c}.$$

The journey will be quickest if the spacecraft accelerates at g *until the halfway point, then decelerates at* g *to come to rest at the Galactic centre, and then does the reverse. So we want to calculate the time taken to get to the half-way mark for which* $T = 12{,}500$ *years, and then multiply the answers by a factor 4. This gives*

$$\tau = 39.4\,\textit{years}.$$

The time taken according to the Earth-based observer comes from

$$t = \frac{c}{a}\sinh\frac{a\tau}{c},$$

which with a bit of work can be shown to be

$$t = \left(T^2 + 2\frac{cT}{g}\right)^{1/2} \approx T + \frac{c}{g}.$$

$c/g = 0.97$ years, so the total time according to Earth is $\approx 50{,}004$ years.

In other words the time taken to approach the speed of light adds surprisingly little to the round-trip time compared to the 50,000 years that light would take, and it is feasible in principle to get to the centre of the Galaxy and back within a human lifetime (as far as the astronauts are concerned) without requiring unpleasantly large accelerations.

(i) * The spacecraft of the previous part is powered by a "photon drive" in which matter/anti-matter annihilation produces gamma-rays which are collimated and sent into space to provide the acceleration (…if only!). Calculate the minimum mass of the rocket when it leaves Earth required to end up with mass m on its return.

In the IRF, if the mass changes from m to $m + dm$, while the speed changes from 0 to dv due to the emission of photons, then conserving momentum

$$0 = (m + dm)dv + cdm,$$

so

$$\frac{dm}{d\tau} = -\frac{gm}{c}.$$

Therefore

$$m = m_0 e^{-g\tau/c}.$$

This reduction factor is suffered 4 times over, and the end result is that $m = 10^{-17.6} m_0$. Thus to arrive back with a 1 ton spacecraft would require a bit of a monster of mass $m_0 = 4 \times 10^{20}\,\mathrm{kg}$ at launch. When the engines first power up they would have a power roughly 3000 times that of the Sun and would make short work of vapourising Earth … not a very practical possibility.

1.8. In a frame S an observer has four-velocity $\vec{U}$ while a particle has four-momentum $\vec{P}$. Show that the energy of the particle E that the *observer* would measure (i.e. not the energy as measured in S) is given by

$$E = \vec{P} \cdot \vec{U}$$

In the rest frame of the observer $\vec{U} = (c, 0, 0, 0)$ while $\vec{P} = (E/c, \mathbf{p})$. Hence the equation given is obviously true in this frame, and since it is manifestly Lorentz invariant, it is true in all other frames.

2.1. Use the momentum four-vector $\vec{P} = m\vec{U}$ where $\vec{U}$ is the four-velocity to obtain the well-known formula $E^2 - p^2c^2 = m^2c^4$ where E is the energy and p the momentum of a particle of (rest) mass m.

The four-momentum can be written as $(E/c, \mathbf{p})$ *so that the invariant norm is then*

$$\vec{P} \cdot \vec{P} = \eta_{\alpha\beta} P^\alpha P^\beta = (E/c)^2 - p^2.$$

The actual value is easiest to calculate in the rest frame of the particle in which $E = mc^2$, $\mathbf{p} = 0$*, and the result follows.*

2.2. The four-momentum of a photon of angular frequency ω and wave-vector $\mathbf{k}$ is given by $\vec{P} = \hbar(\omega/c, \mathbf{k})$.

(a) What sort of vector (timelike, spacelike, null) is $\vec{P}$?

$$\vec{P} \cdot \vec{P} = \eta_{\alpha\beta} P^\alpha P^\beta = \hbar((\omega/c)^2 - k^2) = 0,$$

so it is a null vector (since $c = \omega/k$*), as it should be since it points along the photon's worldline. Note that one cannot define a four-velocity for photons in the same way as for massive particles since* $d\tau = 0$ *for photons.*

(b) Obtain an expression for the Lorentz scalar $\vec{X} \cdot \vec{P}$ where $\vec{X} = (x^0, x^1, x^2, x^3)$. What is its physical interpretation?

$$\vec{X} \cdot \vec{P} = \eta_{\alpha\beta} X^\alpha P^\beta = \hbar(\omega t - \mathbf{k} \cdot \mathbf{x}),$$

remembering that $x^0 = ct$*. This quantity is the phase of the wave, which one expects to be invariant since e.g. a node of an EM wave in one frame where the electric and magnetic fields are zero should also be a node in any other frame and the location of nodes is defined by the phase.*

(c) A photon travels in the x–y plane in a frame S at angle θ measured anti-clockwise from the x-axis. Show that, in a frame S' moving relative to S at speed v in the positive x-direction, the angle is measured as θ' where

$$\cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta},$$

where $\beta = v/c$.

Apply the Lorentz transform to the wavevector $(\omega/c, k\cos\theta, k\sin\theta, 0)$. In the standard orientation defines, $k_{z'} = k_z = 0$ so the photon will travel in the x'–y' plane and the wavevector in S' can therefore be written $(\omega'/c, k'\cos\theta', k\sin\theta', 0)$, and therefore

$$\begin{pmatrix} \omega'/c \\ k'\cos\theta' \\ k'\sin\theta' \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \omega/c \\ k\cos\theta \\ k\sin\theta \\ 0 \end{pmatrix}.$$

Therefore

$$\begin{aligned} \frac{\omega'}{c} &= \gamma\left(\frac{\omega}{c} - \beta k\cos\theta\right), \\ k'\cos\theta' &= \gamma\left(k\cos\theta - \beta\frac{\omega}{c}\right), \\ k'\sin\theta' &= k\sin\theta. \end{aligned}$$

Remembering that $\omega/k = \omega'/k' = c$ (speed of light constant), and dividing the second equation by the first we get

$$\begin{aligned} \cos\theta' &= \frac{k\cos\theta - \beta\omega/c}{\omega/c - \beta k\cos\theta}, \\ &= \frac{\cos\theta - \beta}{1 - \beta\cos\theta}. \end{aligned}$$

The change in angle is known as "aberration" and causes a ± 20 arcsecond variation in the positions of astronomical objects as Earth orbits the Sun first found by James Bradley in 1725.

2.3. Which, if any, of the following are valid tensor relations?

(a) $A^\alpha + B_\alpha$

Invalid. Cannot add tensors with differing numbers of covariant and contravariant indices since they don't transform in the same way.

(b) $R^\alpha{}_\beta A^\beta + B^\alpha = 0$

Valid.

(c) $R_{\alpha\beta} = T_\gamma$

Invalid. Two indices on the left, one on the right means that the left and right cannot transform in the same way and the indices don't match at all either.

(d) $A_{\alpha\beta} = B_{\beta\alpha}$

Valid. Indices need not appear in the same horizontal order.

2.4. Prove by applying the transformation rules that, if P_α and V^α are components of tensors, then $P_\alpha V^\alpha$ is a scalar.

The individual parts transform as follows

$$\begin{aligned} P_{\alpha'} &= \Lambda^{\beta}{}_{\alpha'} P_\beta, \\ V^{\alpha'} &= \Lambda^{\alpha'}{}_{\gamma} V^\gamma, \end{aligned}$$

therefore

$$P_{\alpha'} V^{\alpha'} = \Lambda^{\beta}{}_{\alpha'} \Lambda^{\alpha'}{}_{\gamma} P_\beta V^\gamma.$$

The two LTs represent the transform from S to S' (second one) followed by S' to S (first one), which gives the identity δ^β_γ, so

$$P_{\alpha'} V^{\alpha'} = \delta^\beta_\gamma P_\beta V^\gamma = P_\beta V^\beta.$$

QED.

2.5. Just like vectors, one-forms have a coordinate-independent meaning, so a given one-form $\tilde{p}$ can be written in frames S and S' as $\tilde{p} = p_\alpha \tilde{\omega}^\alpha = p_{\alpha'} \tilde{\omega}^{\alpha'}$. Use this and reasoning similar to that of the lectures to deduce that the basis one-forms transform as

$$\tilde{\omega}^{\alpha'} = \Lambda^{\alpha'}{}_{\beta} \, \tilde{\omega}^\beta.$$

We know that

$$p_\alpha = \Lambda^{\beta'}{}_{\alpha} p_{\beta'},$$

so

$$\Lambda^{\beta'}{}_{\alpha} p_{\beta'} \tilde{\omega}^\alpha = p_{\alpha'} \tilde{\omega}^{\alpha'}.$$

Re-labelling on the left-hand side, $\beta' \to \alpha'$, $\alpha \to \beta$, and collecting up coefficients of $p_{\alpha'}$ we get

$$\left(\tilde{\omega}^{\alpha'} - \Lambda^{\alpha'}{}_{\beta} \tilde{\omega}^\beta \right) p_{\alpha'} = 0.$$

For this to be true for arbitrary $p_{\alpha'}$, the term in brackets must be zero, QED.

2.6. Tensors are linear in all of their arguments so that e.g.

$$T(\alpha\vec{A} + \beta\vec{B}, \tilde{p}) = \alpha T(\vec{A}, \tilde{p}) + \beta T(\vec{B}, \tilde{p}),$$

and

$$T(\vec{A}, \alpha\tilde{p} + \beta\tilde{q}) = \alpha T(\vec{A}, \tilde{p}) + \beta T(\vec{A}, \tilde{q}),$$

where α and β are constants.

Use these and the definition of tensor components from lectures to show that if $\vec{v} = v^\alpha \vec{e}_\alpha$ and $\tilde{p} = p_\alpha \tilde{\omega}^\alpha$ then

$$T(\vec{v}, \tilde{p}) = T_\alpha{}^\beta v^\alpha p_\beta,$$

$$\begin{aligned} T(\vec{v}, \tilde{p}) &= T(v^\alpha \vec{e}_\alpha, p_\beta \tilde{\omega}^\beta), \\ &= v^\alpha T(\vec{e}_\alpha, p_\beta \tilde{\omega}^\beta), \\ &= v^\alpha p_\beta T(\vec{e}_\alpha, \tilde{\omega}^\beta), \\ &= v^\alpha p_\beta T_\alpha{}^\beta, \end{aligned}$$

QED. Here line 2 follows from line 1 by the first relation given, while the next line follows from the second relation. The final line follows from the definition of tensor components given in lectures.

2.7. There is no need to distinguish between "covariant" and "contravariant" indices in Cartesian coordinates, which is why until now you will rarely, if ever, have encountered one-forms. Show that in such coordinates the one-form $\tilde{A}$ dual to the vector $\vec{A}$ has components given by $A_\alpha = A^\alpha$. [Hence Cartesian tensors are usually written entirely with subscripted indices.]

The line element in Cartesian coordinates is given by $dl^2 = dx^2 + dy^2 + dz^2$, so the metric becomes $\eta_{\alpha\beta} = \delta_{\alpha\beta}$, therefore applying the index-lowering property of the metric tensor

$$A_\alpha = \eta_{\alpha\beta} A^\beta = \delta_{\alpha\beta} A^\beta.$$

The $\delta_{\alpha\beta}$ simply selects terms for which $\beta = \alpha$, and so $A_\alpha = A^\alpha$.

2.8. Write down an expression involving the metric tensor that converts a tensor from the form $T^\alpha{}_{\beta\gamma}{}^\delta$ into the form $T_\alpha{}^{\beta\gamma}{}_\delta$.

$$T_\alpha{}^{\beta\gamma}{}_\delta = \eta_{\alpha\mu} \eta^{\beta\nu} \eta^{\gamma\rho} \eta_{\delta\sigma} T^\mu{}_{\nu\rho}{}^\sigma.$$

2.9. As $\eta_{\alpha\beta}$ is a tensor, it must obey the transformation rule

$$\eta_{\alpha'\beta'} = \Lambda^\gamma{}_{\alpha'} \Lambda^\delta{}_{\beta'} \eta_{\gamma\delta}.$$

By writing out this transform as a product of matrices show that the usual Lorentz transform from frame S to frame S' moving at speed v in the x-direction relative to S satisfies this equation.

The right-hand side involves two transforms from S' to S. In matrix form one pre-multiplies the metric while the other post-multiplies since the summations are over the two indices of the metric. Thus the transformation can be written

$$\begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Multiplying the second two matrices leaves

$$\begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ -\gamma\beta & -\gamma & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and carrying out the final multiplication, using $\gamma^2 - \gamma^2\beta^2 = 1$, gives

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

as anticipated.

2.10. Prove that the Kronecker delta, δ^α_β, transforms as a tensor.

Consider

$$\Lambda^{\alpha'}{}_\mu \Lambda^\nu{}_{\beta'} \delta^\mu_\nu.$$

The Kronecker delta selects terms for which its two indices match and so this quantity is easily seen to be equal to

$$\Lambda^{\alpha'}{}_\mu \Lambda^\mu{}_{\beta'}.$$

Reading from right-to-left, this corresponds to an LT from S' to S followed by one from S to S', and so is the identity transform from S' to S' which is expressible as $\delta^{\alpha'}_{\beta'}$ (also easily shown with matrices). Therefore

$$\delta^{\alpha'}_{\beta'} = \Lambda^{\alpha'}{}_\mu \Lambda^\nu{}_{\beta'} \delta^\mu_\nu,$$

which is the transformation law of a $(1,1)$ tensor.

2.11. Why does the gradient vector of a scalar ϕ take the form $(\partial\phi/\partial x^0, -\partial\phi/\partial x^1, -\partial\phi/\partial x^2, -\partial\phi/\partial x^3)$ in SR?

Because the components $\partial\phi/\partial x^\alpha$ transforms as a one-form, not a vector. If we use the index-raising property of the metric (which in SR simply reverses the signs of the spatial components) we arrive at the vector version given.

2.12. Work out the values of the components of the fully contra-variant form of the Kronecker-delta, $\delta^{\alpha\beta}$ in the standard coordinates of SR.

Raising the index

$$\delta^{\alpha\beta} = \eta^{\alpha\gamma}\delta^{\beta}_{\gamma} = \eta^{\alpha\beta}.$$

So, slightly unexpectedly perhaps, the contravariant Kronecker-delta is diagonal with values $(1, -1, -1, -1)$, *and since it is the same as the metric, you won't see it.*

2.13. The "exterior product" of two vectors $\vec{A}$ and $\vec{B}$ is written as $\vec{A} \otimes \vec{B}$ and is defined to be the tensor that when applied to two arbitrary one-forms, $\tilde{p}$ and $\tilde{q}$, returns the product of $\vec{A}$ and $\vec{B}$ operated on each input argument separately, i.e. if $T = \vec{A} \otimes \vec{B}$ then

$$T(\tilde{p}, \tilde{q}) = \vec{A}(\tilde{p})\vec{B}(\tilde{q}).$$

Show that the components of T are $T^{\alpha\beta} = A^\alpha B^\beta$.

T has components given by feeding it basis one-forms:

$$T(\tilde{\omega}^\alpha, \tilde{\omega}^\beta) = \vec{A}(\tilde{\omega}^\alpha)\vec{B}(\tilde{\omega}^\beta) = A^\alpha B^\beta,$$

QED.

If $T(\tilde{p}, \tilde{q}) = T(\tilde{q}, \tilde{p})$ for any $\tilde{p}$ and $\tilde{q}$, what condition must $\vec{A}$ and $\vec{B}$ satisfy?

In components

$$A^\alpha B^\beta p_\alpha q_\beta = A^\gamma B^\delta q_\gamma p_\delta.$$

Re-labelling δ to α on the right-hand side, and gathering coefficients of p_α:

$$\left(A^\alpha B^\beta q_\beta - A^\gamma B^\alpha q_\gamma\right) p_\alpha = 0,$$

which since p_α is arbitrary implies that

$$A^\alpha B^\beta q_\beta = A^\gamma B^\alpha q_\gamma.$$

A similar operation applied to q_γ *implies that*

$$A^\alpha B^\beta = A^\beta B^\alpha,$$

which it can be seen implies that $\vec{A} = k\vec{B}$ *where* k *is a constant.*

Can *any* tensor with two contravariant indices, $T^{\alpha\beta}$, be written as the exterior product of two vectors?

No. An arbitrary 2-index tensor has N^2 *components while the vectors can only supply* $2N$ *such components, so for any* $N > 2$*, it is not possible.*

2.14. Show that if the stress-energy tensor T has components in the fluid's rest frame (IRF)

$$T^{\alpha\beta} = \begin{pmatrix} \rho_0 c^2 & 0 & 0 & 0 \\ 0 & p_0 & 0 & 0 \\ 0 & 0 & p_0 & 0 \\ 0 & 0 & 0 & p_0 \end{pmatrix},$$

where ρ_0 and p_0 are the fluid's rest frame density and pressure then it can be written as

$$T^{\alpha\beta} = \left(\rho_0 + \frac{p_0}{c^2}\right) U^\alpha U^\beta - p_0 \eta^{\alpha\beta}, \tag{1}$$

and also as

$$T = \left(\rho_0 + \frac{p_0}{c^2}\right) \vec{U} \otimes \vec{U} - p_0 \eta,$$

where η is the metric tensor.

The four-velocity $\vec{U} = \gamma(c, \mathbf{v})$*, which in the rest frame is* $(c, 0, 0, 0)$*. Hence*

$$T^{00} = \left(\rho_0 + \frac{p_0}{c^2}\right) U^0 U^0 - p_0 \eta^{00} = \rho_0 c^2,$$

since $U^0 = c$ *and* $\eta^{00} = 1$*. Similarly*

$$T^{11} = \left(\rho_0 + \frac{p_0}{c^2}\right) U^1 U^1 - p_0 \eta^{11} = p,$$

since $\eta^{11} = -1$ *and* $U^1 = 0$*. In a similar fashion, all components can be verified. The important point is that the new expression is clearly a tensor as it is built up of scalars, vectors and tensors, and is therefore true in ANY frame.*

The final expression follows on considering a scalar contraction such as

$$T(\tilde{p}, \tilde{q}) = T^{\alpha\beta} p_\alpha q_\beta.$$

The first part gives expressions of the form

$$U^\alpha U^\beta p_\alpha q_\beta = \vec{U}(\tilde{p}) \vec{U}(\tilde{q}),$$

and comparing with the previous expression, the $\vec{U} \otimes \vec{U}$ *term is obvious.*

2.15. By applying the appropriate transformation of the IRF components, calculate the components of the stress-energy tensor of a perfect fluid in a frame in which the fluid is travelling at speed v in the positive x direction.

Verify that your expression agrees with Eq. 1.

Try to interpret your results for the case of non-relativistic fluids.

Recall the transformation of tensors:

$$T^{\alpha'\beta'} = \Lambda^{\alpha'}{}_{\alpha}\Lambda^{\beta'}{}_{\beta}T^{\alpha\beta}.$$

$\Lambda^{\alpha'}{}_{\alpha}$ *is the Lorentz transform that takes us from frame* S *to* S'*, and in this case* S' *is moving at* v *is the negative* x *direction relative to* S *so*

$$\left[\Lambda^{\alpha'}{}_{\alpha}\right] = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

using square brackets to mean the matrix equivalent to the set of components they enclose. Therefore:

$$\left[T^{\alpha'\beta'}\right] = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho_0 c^2 & 0 & 0 & 0 \\ 0 & p_0 & 0 & 0 \\ 0 & 0 & p_0 & 0 \\ 0 & 0 & 0 & p_0 \end{pmatrix} \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

*NB As in Q**2.9**, In matrix terms the second LT must come second because the summation over* β *selects columns. Multiplying through the matrices gives*

$$\left[T^{\alpha'\beta'}\right] = \begin{pmatrix} \gamma^2\rho_0 c^2 + \gamma^2\beta^2 p_0 & \gamma^2\beta\rho_0 c^2 + \gamma^2\beta p_0 & 0 & 0 \\ \gamma^2\beta\rho_0 c^2 + \gamma^2\beta p_0 & \gamma^2\beta^2\rho_0 c^2 + \gamma^2 p_0 & 0 & 0 \\ 0 & 0 & p_0 & 0 \\ 0 & 0 & 0 & p_0 \end{pmatrix}.$$

Setting $\vec{U} = \gamma(c, v, 0, 0)$ *in Eq. 1 with* $\alpha = \beta = 0$,

$$\begin{aligned} T^{00} &= \left(\rho_0 + \frac{p_0}{c^2}\right) U^0 U^0 - p_0 \eta^{00}, \\ &= \left(\rho_0 + \frac{p_0}{c^2}\right) \gamma^2 c^2 - p_0, \\ &= \gamma^2 \rho_0 c^2 + (\gamma^2 - 1) p_0, \\ &= \gamma^2 \rho_0 c^2 + \gamma^2 \beta^2 p_0. \end{aligned}$$

The other components follow in the same manner.

At non-relativistic speeds and pressures, $p_0 \ll \rho_0 c^2$, $\gamma \approx 1$ *and* $\beta \ll 1$*, and keeping only the dominant terms of each component, the stress-energy tensor reduces to*

$$\left[T^{\alpha'\beta'}\right] \approx \begin{pmatrix} \rho_0 c^2 & v\rho_0 c & 0 & 0 \\ v\rho_0 c & p_0 + \rho_0 v^2 & 0 & 0 \\ 0 & 0 & p_0 & 0 \\ 0 & 0 & 0 & p_0 \end{pmatrix}.$$

The T^{11} component, representing the flux of x-momentum across a surface of constant x acquires a $\rho_0 v^2$ "ram pressure" term. The T^{01} components are essentially mass fluxes in the x direction.

2.16. The conservation of energy and momentum in relativity is expressed by:

$$T^{\alpha\beta}{}_{,\beta} = 0.$$

(a) How many equations does this expression contain?

4, one for each value of α.

(b) Starting from Eq. 1, show that for $\alpha = 0$ the conservation equation leads to:

$$\frac{\partial}{\partial t}\left[\gamma^2\left(\rho_0 c^2 + \beta^2 p_0\right)\right] + \nabla\cdot\left[\gamma^2\left(\rho_0 c^2 + p_0\right)\mathbf{v}\right] = 0.$$

Using Eq. 1, $X^0 = ct$, $U^0 = \gamma c$, $\eta^{00} = 1$ and $\eta^{0i} = 0$,

$$T^{0\beta}{}_{,\beta} = \frac{\partial}{\partial ct}\left(\rho_0 + \frac{p_0}{c^2}\right)\gamma^2 c^2 + \frac{\partial}{\partial x^i}\left(\rho_0 + \frac{p_0}{c^2}\right)\gamma c U^i - \frac{\partial}{\partial ct} p_0 = 0.$$

Multiplying through by c and collecting terms

$$\frac{\partial}{\partial t}\left[\gamma^2\left(\rho_0 c^2 + p_0\right) - p_0\right] + \nabla\cdot\left[\gamma^2\left(\rho_0 c^2 + p_0\right)\mathbf{v}\right] = 0.$$

Substituting $\gamma^2 - 1 = \gamma^2\beta^2$, the result follows.

(c) Write down the Newtonian analogue of the equation of the previous part.

The continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0.$$

2.17. * Use the relativistic conservation relations $T^{\alpha\beta}{}_{,\beta} = 0$ to prove the following three results which are useful in the theory of gravitational waves:

$$\begin{aligned}
\frac{\partial}{\partial t}\int T^{\alpha 0}\,dV &= 0,\\
\int T^{ij}\,dV &= \frac{1}{c}\frac{\partial}{\partial t}\int T^{i0}x^j\,dV,\\
\int T^{ij}\,dV &= \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int T^{00}x^i x^j\,dV.
\end{aligned}$$

In these expression dV indicates a volume integral over finite distributions of matter and it may be assumed that $T^{\alpha\beta} = 0$ outside these distributions.

You will need the following generalisations of Gauss' theorem

$$\begin{aligned} \int T^{\alpha k}{}_{,k}\, dV &= \oint T^{\alpha k} n_k\, dS, \\ \int \left(T^{\alpha k} x^i\right)_{,k} dV &= \oint T^{\alpha k} n_k x^i\, dS, \\ \int \left(T^{\alpha k} x^i x^j\right)_{,k} dV &= \oint T^{\alpha k} n_k x^i x^j\, dS. \end{aligned}$$

where the n_i are components of the unit 3-vectors pointing out of a surface S enclosing a volume V, and as usual Latin indices i, j, k indicate the spatial components alone.

Choosing a surface which surrounds the matter such that $T^{\alpha\beta} = 0$ over the whole surface it is evident that each of the surface integrals on the right of Gauss' theorem can be made to be zero, and thus each of the volume integrals is too. Considering the first one we have

$$\int T^{\alpha k}{}_{,k}\, dV = 0.$$

Now the energy–momentum conservation relations can be written as

$$T^{\alpha\beta}{}_{\beta} = T^{\alpha 0}{}_{,0} + T^{\alpha i}{}_{,i} = 0.$$

Integrating over the volume, the second term on the right drops out because of the result above and we are left with

$$\int T^{\alpha 0}{}_{,0}\, dV = 0.$$

Since the 0 index implies the time component, ct, and swapping the order of integration and differentiation, we have

$$\frac{\partial}{\partial t} \int T^{\alpha 0}\, dV = 0,$$

which is the first result. This expresses the conservation of mass and momentum for an object subject to no external momentum and energy transfer. The second version of Gauss' theorem similarly implies

$$\int \left(T^{\alpha k} x^i\right)_{,k} dV = 0.$$

Expanding out the derivative implies that

$$\int T^{\alpha k}{}_{,k}\, x^i\, dV + \int T^{\alpha i}\, dV = 0,$$

where the relation $x^i{}_{,k} = \delta^i_k$ has been used. The first term can be transformed as above and we obtain

$$\int T^{\alpha i}\, dV = \int T^{\alpha 0}{}_{,0}\, x^i\, dV. \tag{2}$$

Taking the time derivative outside the right-hand integral with $x^0 = ct$, and specialising and re-labelling $\alpha \to i$, $i \to j$, the second result is obtained.

Finally, applying a similar expansion to the integral on the left of the third version of Gauss' theorem gives

$$\int T^{\alpha k}{}_{,k}\, x^i x^j \, dV + \int T^{\alpha i} x^j \, dV + \int T^{\alpha j} x^i \, dV = 0.$$

Using the conservation equations to transform the first term we find

$$\int T^{\alpha i} x^j \, dV + \int T^{\alpha j} x^i \, dV = \int T^{\alpha 0}{}_{,0}\, x^i x^j \, dV.$$

Taking the time derivative of this equation and moving the derivative outside the integral on the right-hand side gives

$$\int T^{\alpha i}{}_{,0} x^j \, dV + \int T^{\alpha j}{}_{,0} x^i \, dV = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int T^{\alpha 0}\, x^i x^j \, dV.$$

Setting $\alpha = 0$, and remembering the symmetry of the stress-energy tensor, the two left-hand integrals can be transformed using Eq. 2 to give

$$\int T^{ij} \, dV + \int T^{ji} \, dV = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int T^{00}\, x^i x^j \, dV.$$

The final result, which is known as the tensor virial theorem, immediately follows. For non-relativistic fluids this becomes

$$\int T^{ij} \, dV = \frac{\partial^2}{\partial t^2}\int \rho\, x^i x^j \, dV.$$

The integral on the right-hand side is the "quadrupole-moment" or "moment-of-inertia" tensor.

2.18. A spacecraft moves through a nebula measuring the temperature as a function of time. In a frame in which the nebula is stationary the spacecraft has four-velocity $\vec{U}$. Show that the rate of change of temperature with time measured in the spacecraft is given by

$$\frac{dT}{d\tau} = \nabla T(\vec{U}),$$

i.e. the one-form gradient operating on the four-velocity to produce a scalar.

In the rest frame of the spacecraft $\vec{U} = (c, 0, 0, 0)$ while $\nabla T = ((\partial T/\partial t)/c, \ldots)$, the spatial components being irrelevant. Operating with the gradient (a one-form) on the vector gives

$$\nabla T(\vec{U}) = \frac{\partial T}{\partial t} = \frac{dT}{d\tau}.$$

since in the rest frame of the spacecraft $t = \tau$ and with no change in spatial coordinate the partial derivative becomes an ordinary derivative. The usual argument about covariance then makes this a result that applies in all frames.

Write out the right hand side of this equation in full and interpret the resulting expression.

Setting $\vec{U} = \gamma(c, v_x, v_y, v_z)$ *and*

$$\nabla T = \left(\frac{1}{c}\frac{\partial T}{\partial t}, \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right),$$

gives

$$\frac{dT}{d\tau} = \gamma\left(\frac{\partial T}{\partial t} + (\mathbf{v}\cdot\nabla)\,T\right).$$

This can be understood to be made up of a term due to the change in temperature of the nebula at any point with time plus a term due to the spacecraft's motion through regions of spatially variable temperature. The γ *factor comes from time dilation. Note how neatly the 4-vector formalism handles this case.*