Department of Physics
University of Warwick
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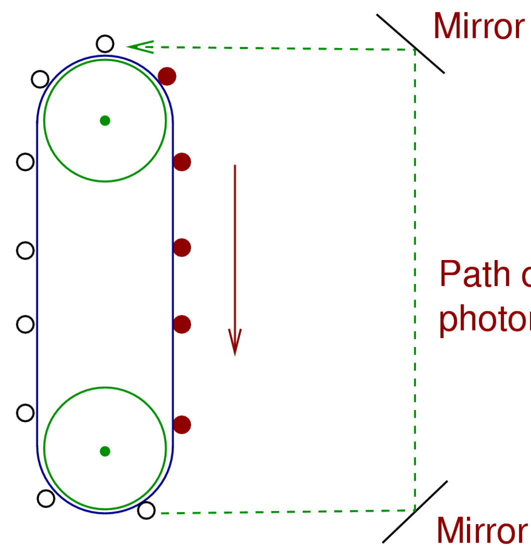
PX436, General Relativity Problems

*Please have a good go at the problems. They are a very important part of the learning for this course, and you should try to devote around 5 to 7 hours/week to them of the 15 hours/week expected for this course (3 hours for the lectures, 5 to 7 for reading). It is important to give problems your best shot before turning to the hints or answers. Since they are effectively "open book" they are often more difficult than corresponding exam questions. You should expect to need your notes and books to complete them: that's part of the point. The problems follow the sequence of the course. Please also look at the past exam papers for other problems. I will go through selected ones of these during the problems classes. The problems are numbered so that e.g. "2.3" is the third question of the material of week 2. *Starred parts are either off the main track of the course, particularly tough, or both!*

1.1. Calculate the radius at which the bending of light as calculated from the equivalence principle would keep it in a circular orbit around a mass M.

1.2. The diagram below shows the design of a perpetual motion machine sent by an inventor called H.Bondi to a patent office in Switzerland where a certain A.Einstein works:



Atoms attached to a movable belt absorb photons at the top of their travel and emit them at the bottom, with the photons directed back to the top-most atoms where they are absorbed. When an atom absorbs a photon of energy $E = h\nu$ its mass increases by $\Delta m = h\nu/c^2$ hence the excited atoms (filled circles) on the right of the belt always outweigh the de-excited atoms (empty circles) on the left and hence perpetual clockwise motion results which can solve all the world's energy problems . . .

What flaw does Einstein spot in this machine in addition to the evident violation of the first law of thermodynamics?

1.3. Why can there be no equivalent of Einstein's equivalence principle for electric rather than gravitational fields?

1.4. It is important to be confident with the summation convention and index manipulation. Here are a few practice problems:

(a) Show that

$$A^\alpha B_\alpha = A^\beta B_\beta.$$

(b) What is the value of δ^α_α?

(c) Show that in SR

$$\vec{A}\cdot\vec{A} = \left(A^0\right)^2 - A^i A^i,$$

where $\vec{A}$ is an arbitrary four vector and in this case the summation convention applies to the last term even though both indices are raised.

(d) If $A_{\alpha\beta}$ is symmetric ($A_{\alpha\beta} = A_{\beta\alpha}$), and $B^{\alpha\beta}$ is anti-symmetric ($B^{\alpha\beta} = -B^{\beta\alpha}$), show that $A_{\alpha\beta}B^{\alpha\beta} = 0$.

(e) Show that, if $A_{\alpha\beta}$ is symmetric but $B^{\alpha\beta}$ is arbitrary, then

$$A_{\alpha\beta}B^{\alpha\beta} = A_{\alpha\beta}C^{\alpha\beta},$$

where

$$C^{\alpha\beta} = \frac{1}{2}\left(B^{\alpha\beta} + B^{\beta\alpha}\right).$$

(f) How many independent components are needed to specify $A^{\alpha\beta}$ when it is (i) arbitrary, (ii) symmetric and (iii) anti-symmetric in 3, 4 and N dimensions?

(g) The "Christoffel symbols", $\Gamma^\alpha{}_{\rho\sigma}$, are symmetric in the indices ρ and σ. How many independent components are there in four dimensions?

1.5. If t and x transform according to

$$\begin{aligned} ct' &= \alpha(ct) + \beta x, \\ x' &= \gamma(ct) + \delta x, \end{aligned}$$

where α, β, γ and δ are constants, such that the interval $s^2 = (ct)^2 - x^2$ is preserved, show that $\alpha^2 - \beta^2 = 1$, $\gamma = \beta$ and $\delta = \alpha$.

Show that these relations are consistent with standard LTs.

1.6. Use the component transformation definition of vectors to show that if $\vec{A}$ and $\vec{B}$ are four-vectors then the quantities V^α defined by $V^\alpha = A^\alpha + B^\alpha$ also transform as a four-vector enabling one to write $\vec{V} = \vec{A} + \vec{B}$.

1.7. The four-velocity in Special Relativity (SR), $\vec{U}$, has components

$$U^\alpha = \frac{dx^\alpha}{d\tau} = \gamma(c, \vec{v}),$$

where in the first representation x^α represent the coordinates of events in the usual representation with $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$, and τ is the proper time, while in the second, γ is the Lorentz factor and $\vec{v}$ the three-velocity.

(a) Why, given the properties of dx^α and τ, is $\vec{U}$ "clearly" a four-vector?

(b) Show that in *any* reference frame

$$\vec{U} \cdot \vec{U} = \eta_{\alpha\beta} U^\alpha U^\beta = c^2,$$

where $\eta_{\alpha\beta}$ is the SR metric as defined in lectures.

(c) Hence show that

$$\vec{U} \cdot \vec{A} = 0,$$

in all frames, where the four-acceleration $\vec{A}$ is defined by

$$A^\alpha = \frac{dU^\alpha}{d\tau}.$$

(d) Show that $\vec{A}$ is given by

$$\vec{A} = \gamma(\dot{\gamma}c, \dot{\gamma}\vec{v} + \gamma\vec{a}),$$

where $\vec{a}$ is the 3-acceleration and the dots indicate differentiation with respect to t.

(e) An object undergoes acceleration a in the x-direction as measured in its instantaneous rest frame (IRF). Show that in a frame in which it travels with speed v in the x-direction, it has an acceleration

$$a' = \gamma^{-3} a,$$

where $\gamma = \gamma(v)$ is the Lorentz factor.

(f) Starting from rest with $v = t = \tau = 0$ and $x = c^2/a$, an object undergoes constant acceleration a in the x-direction in its IRF. Integrate the result of part (**1.7**e) to show that:

$$\begin{aligned} v &= c \tanh \frac{a\tau}{c}, \\ \gamma &= \cosh \frac{a\tau}{c}, \\ x &= \frac{c^2}{a} \cosh \frac{a\tau}{c}, \\ t &= \frac{c}{a} \sinh \frac{a\tau}{c}, \end{aligned}$$

where τ is the proper time, i.e the time measured by clocks travelling with the object.

(g) From the previous part, obtain a single equation relating x and t and hence draw the worldline of the object in a spacetime diagram.

In what sense does the line $x = ct$ act as a "horizon" for the object?

(h) * Use the result of the previous part to calculate the shortest time it would take a spacecraft to travel to the centre of our Galaxy 25,000 light-years from Earth and back again assuming that it could maintain a constant IRF acceleration ("proper" acceleration) of one g at all times and that it comes to rest, however briefly, at the Galactic centre. Calculate the time as reckoned by an observer on Earth and on the spacecraft.

(i) * The spacecraft of the previous part is powered by a "photon drive" in which matter/anti-matter annihilation produces gamma-rays which are collimated and sent into space to provide the acceleration (...if only!). Calculate the minimum mass of the rocket when it leaves Earth required to end up with mass m on its return.

1.8. In a frame S an observer has four-velocity $\vec{U}$ while a particle has four-momentum $\vec{P}$. Show that the energy of the particle E that the *observer* would measure (i.e. not the energy as measured in S) is given by

$$E = \vec{P} \cdot \vec{U}$$

2.1. Use the momentum four-vector $\vec{P} = m\vec{U}$ where $\vec{U}$ is the four-velocity to obtain the well-known formula $E^2 - p^2c^2 = m^2c^4$ where E is the energy and p the momentum of a particle of (rest) mass m.

2.2. The four-momentum of a photon of angular frequency ω and wave-vector $\mathbf{k}$ is given by $\vec{P} = \hbar(\omega/c, \mathbf{k})$.

(a) What sort of vector (timelike, spacelike, null) is $\vec{P}$?

(b) Obtain an expression for the Lorentz scalar $\vec{X} \cdot \vec{P}$ where $\vec{X} = (x^0, x^1, x^2, x^3)$. What is its physical interpretation?

(c) A photon travels in the x–y plane in a frame S at angle θ measured anti-clockwise from the x-axis. Show that, in a frame S' moving relative to S at speed v in the positive x-direction, the angle is measured as θ' where

$$\cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta},$$

where $\beta = v/c$.

2.3. Which, if any, of the following are valid tensor relations?

(a) $A^\alpha + B_\alpha$

(b) $R^\alpha{}_\beta A^\beta + B^\alpha = 0$

(c) $R_{\alpha\beta} = T_\gamma$

(d) $A_{\alpha\beta} = B_{\beta\alpha}$

2.4. Prove by applying the transformation rules that, if P_α and V^α are components of tensors, then $P_\alpha V^\alpha$ is a scalar.

2.5. Just like vectors, one-forms have a coordinate-independent meaning, so a given one-form $\tilde{p}$ can be written in frames S and S' as $\tilde{p} = p_\alpha \tilde{\omega}^\alpha = p_{\alpha'} \tilde{\omega}^{\alpha'}$. Use this and reasoning similar to that of the lectures to deduce that the basis one-forms transform as

$$\tilde{\omega}^{\alpha'} = \Lambda^{\alpha'}{}_\beta \, \tilde{\omega}^\beta.$$

2.6. Tensors are linear in all of their arguments so that e.g.

$$T(\alpha\vec{A} + \beta\vec{B}, \tilde{p}) = \alpha T(\vec{A}, \tilde{p}) + \beta T(\vec{B}, \tilde{p}),$$

and

$$T(\vec{A}, \alpha\tilde{p} + \beta\tilde{q}) = \alpha T(\vec{A}, \tilde{p}) + \beta T(\vec{A}, \tilde{q}),$$

where α and β are constants.

Use these and the definition of tensor components from lectures to show that if $\vec{v} = v^\alpha \vec{e}_\alpha$ and $\tilde{p} = p_\alpha \tilde{\omega}^\alpha$ then

$$T(\vec{v}, \tilde{p}) = T_\alpha{}^\beta v^\alpha p_\beta,$$

2.7. There is no need to distinguish between "covariant" and "contravariant" indices in Cartesian coordinates, which is why until now you will rarely, if ever, have encountered one-forms. Show that in such coordinates the one-form $\tilde{A}$ dual to the vector $\vec{A}$ has components given by $A_\alpha = A^\alpha$. [Hence Cartesian tensors are usually written entirely with subscripted indices.]

2.8. Write down an expression involving the metric tensor that converts a tensor from the form $T^\alpha{}_{\beta\gamma}{}^\delta$ into the form $T_\alpha{}^{\beta\gamma}{}_\delta$.

2.9. As $\eta_{\alpha\beta}$ is a tensor, it must obey the transformation rule

$$\eta_{\alpha'\beta'} = \Lambda^\gamma{}_{\alpha'} \Lambda^\delta{}_{\beta'} \eta_{\gamma\delta}.$$

By writing out this transform as a product of matrices show that the usual Lorentz transform from frame S to frame S' moving at speed v in the x-direction relative to S satisfies this equation.

2.10. Prove that the Kronecker delta, δ^α_β, transforms as a tensor.

2.11. Why does the gradient vector of a scalar ϕ take the form $(\partial\phi/\partial x^0, -\partial\phi/\partial x^1, -\partial\phi/\partial x^2, -\partial\phi/\partial x^3)$ in SR?

2.12. Work out the values of the components of the fully contra-variant form of the Kronecker-delta, $\delta^{\alpha\beta}$ in the standard coordinates of SR.

2.13. The "exterior product" of two vectors $\vec{A}$ and $\vec{B}$ is written as $\vec{A} \otimes \vec{B}$ and is defined to be the tensor that when applied to two arbitrary one-forms, $\tilde{p}$ and $\tilde{q}$, returns the product of $\vec{A}$ and $\vec{B}$ operated on each input argument separately, i.e. if $T = \vec{A} \otimes \vec{B}$ then

$$T(\tilde{p}, \tilde{q}) = \vec{A}(\tilde{p})\vec{B}(\tilde{q}).$$

Show that the components of T are $T^{\alpha\beta} = A^\alpha B^\beta$.

If $T(\tilde{p}, \tilde{q}) = T(\tilde{q}, \tilde{p})$ for any $\tilde{p}$ and $\tilde{q}$, what condition must $\vec{A}$ and $\vec{B}$ satisfy?

Can *any* tensor with two contravariant indices, $T^{\alpha\beta}$, be written as the exterior product of two vectors?

2.14. Show that if the stress-energy tensor T has components in the fluid's rest frame (IRF)

$$T^{\alpha\beta} = \begin{pmatrix} \rho_0 c^2 & 0 & 0 & 0 \\ 0 & p_0 & 0 & 0 \\ 0 & 0 & p_0 & 0 \\ 0 & 0 & 0 & p_0 \end{pmatrix},$$

where ρ_0 and p_0 are the fluid's rest frame density and pressure then it can be written as

$$T^{\alpha\beta} = \left(\rho_0 + \frac{p_0}{c^2}\right) U^\alpha U^\beta - p_0 \eta^{\alpha\beta}, \qquad (1)$$

and also as

$$T = \left(\rho_0 + \frac{p_0}{c^2}\right) \vec{U} \otimes \vec{U} - p_0 \eta,$$

where η is the metric tensor.

2.15. By applying the appropriate transformation of the IRF components, calculate the components of the stress-energy tensor of a perfect fluid in a frame in which the fluid is travelling at speed v in the positive x direction.

Verify that your expression agrees with Eq. 1.

Try to interpret your results for the case of non-relativistic fluids.

2.16. The conservation of energy and momentum in relativity is expressed by:

$$T^{\alpha\beta}{}_{,\beta} = 0.$$

(a) How many equations does this expression contain?

(b) Starting from Eq. 1, show that for $\alpha = 0$ the conservation equation leads to:

$$\frac{\partial}{\partial t}\left[\gamma^2\left(\rho_0 c^2 + \beta^2 p_0\right)\right] + \nabla \cdot \left[\gamma^2\left(\rho_0 c^2 + p_0\right)\mathbf{v}\right] = 0.$$

(c) Write down the Newtonian analogue of the equation of the previous part.

2.17. * Use the relativistic conservation relations $T^{\alpha\beta}{}_{,\beta} = 0$ to prove the following three results which are useful in the theory of gravitational waves:

$$\begin{aligned}
\frac{\partial}{\partial t}\int T^{\alpha 0}\, dV &= 0, \\
\int T^{ij}\, dV &= \frac{1}{c}\frac{\partial}{\partial t}\int T^{i0} x^j\, dV, \\
\int T^{ij}\, dV &= \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int T^{00} x^i x^j\, dV.
\end{aligned}$$

In these expression dV indicates a volume integral over finite distributions of matter and it may be assumed that $T^{\alpha\beta} = 0$ outside these distributions.

You will need the following generalisations of Gauss' theorem

$$\begin{aligned}
\int T^{\alpha k}{}_{,k}\, dV &= \oint T^{\alpha k} n_k\, dS, \\
\int \left(T^{\alpha k} x^i\right)_{,k} dV &= \oint T^{\alpha k} n_k x^i\, dS, \\
\int \left(T^{\alpha k} x^i x^j\right)_{,k} dV &= \oint T^{\alpha k} n_k x^i x^j\, dS.
\end{aligned}$$

where the n_i are components of the unit 3-vectors pointing out of a surface surface S enclosing a volume V, and as usual Latin indices i, j, k indicate the spatial components alone.

2.18. A spacecraft moves through a nebula measuring the temperature as a function of time. In a frame in which the nebula is stationary the spacecraft has four-velocity $\vec{U}$. Show that the rate of change of temperature with time measured in the spacecraft is given by

$$\frac{dT}{d\tau} = \nabla T(\vec{U}),$$

i.e. the one-form gradient operating on the four-velocity to produce a scalar.

Write out the right hand side of this equation in full and interpret the resulting expression.

3.1. A cloud of charges has charge density ρ_0 in a frame in which the charges are stationary. Defining the current density four-vector $\vec{J} = \rho_0 \vec{U}$, where $\vec{U}$ is the four-velocity, one can show that $J^\alpha{}_{,\alpha} = 0$.

(a) Why must $\vec{J}$ be a four-vector?

(b) Now consider a frame in which the charges move with 3-velocity $\mathbf{v}$. Defining $\rho = \gamma\rho_0$ and $\mathbf{j} = \gamma\rho_0\mathbf{v}$, write out $J^\alpha{}_{,\alpha} = 0$ in full.

(c) Give a physical interpretation of the resulting expression.

3.2. The surface of a cylinder radius R with its axis along the z axis can be labelled by the coordinates ϕ and z of the usual cylindrical coordinates r, ϕ, z.

(a) Write down an expression for the interval/line-element ds^2 in these coordinates.

(b) Obtain an expression for the line element in terms of new coordinates x and y where $x = R\phi$ and $y = z$.

(c) In terms of its "interior" geometry, as represented by the line element, is the surface of the cylinder flat or curved?

3.3. In terms of the usual spherical polar angles, θ and ϕ, the line element on the surface of a sphere of radius R can be written

$$ds^2 = R^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right).$$

(a) Defining $r = R\sin\theta$, show that the line element can be re-written in terms of r and θ as

$$ds^2 = \frac{dr^2}{1 - (r/R)^2} + r^2 \, d\phi^2.$$

(b) The line element of part (a) exhibits a singularity at $r = R$. Identify which region of the sphere this corresponds to; a similar "coordinate singularity" occurs in the Schwarzschild metric.

3.4. Figure 1 shows a "stereographic projection" in which a point P on the surface of Earth is projected to P' in a plane tangent at the North pole N along a line running from the South pole S through P. Polar coordinates ρ, ϕ in the tangent plane with N at the origin are then coordinates for the surface of the sphere (except for S).

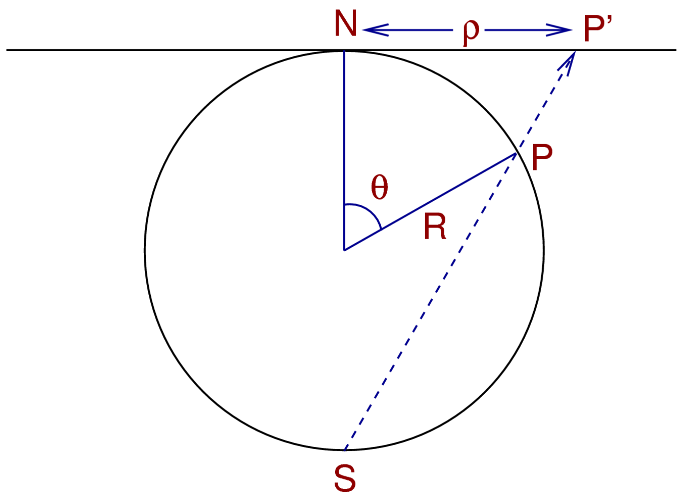


Figure 1: A stereographic projection.

(a) Show that, in terms of the usual spherical polar angle θ, assuming a z axis running from S to N, ρ is given by

$$\rho = 2R\tan(\theta/2),$$

where R is Earth's radius.

(b) Hence, starting from the spherical interval $ds^2 = R^2\,d\theta^2 + R^2\sin^2\theta\,d\phi^2$, show that in terms of coordinates ρ, ϕ the interval is given by

$$ds^2 = \frac{d\rho^2 + \rho^2\,d\phi^2}{\left(1 + (\rho/2R)^2\right)^2}.$$

(c) * The stereographic projection preserves the shapes of small regions on the Earth and is thus said to be "conformal". Can you see why this is from the above line element?

3.5. * The equations of Newtonian fluid mechanics in Cartesian component form (no distinction between co- and contra-variant indices, all subscripted) for inviscid fluids are

$$\frac{\partial\rho}{\partial t} + \nabla_i(\rho v_i) = 0, \tag{2}$$

and

$$\rho\left(\frac{\partial v_i}{\partial t} + v_j\nabla_j v_i\right) = -\nabla_i p, \tag{3}$$

where $\nabla_i = \partial/\partial x^i$, $i = 1$, 2 or 3, ρ is the density, v_i is a component of the (three-)velocity and p is the pressure. (Summation convention still applies to repeated indices.)

Show from Equations 1 and 2 that

$$\frac{\partial\rho v_i}{\partial t} + \nabla_j t_{ij} = 0, \tag{4}$$

where the tensor t_{ij} is given by

$$t_{ij} = \rho v_i v_j + p\delta_{ij}.$$

Equations 1 and 3 express mass and momentum conservation and are the Newtonian analogues of the SR relations $T^{\alpha\beta}{}_{,\beta} = 0$.

3.6. The following problem illustrates a subtlety not covered in the lectures: not all bases can be represented by coordinates. The orthogonal unit vectors commonly used in polar and spherical polar coordinates are examples of such *non-coordinate* bases. Other than this problem, all bases in this course are assumed to be coordinate bases as it simplifies several expressions. However, the last part of the question illustrates one of the resulting pitfalls if you extrapolate too simply from familiar results from the past.

Consider a transformation from 2D Cartesian coordinates (x, y) to new coordinates (u, v). Let the transformation of basis vectors be written as

$$\begin{aligned} \vec{e}_u &= {L_u}^x\, \vec{e}_x + {L_u}^y\, \vec{e}_y, \\ \vec{e}_v &= {L_v}^x\, \vec{e}_x + {L_v}^y\, \vec{e}_y. \end{aligned}$$

(a) By comparing this expression with the general formula for the transformation of basis vectors, show that the coefficients above must obey the following conditions:

$$\begin{aligned} \partial_v {L_u}^x &= \partial_u {L_v}^x, \\ \partial_v {L_u}^y &= \partial_u {L_v}^y. \end{aligned}$$

(b) Show that the usual unit vectors, $\hat{\mathbf{r}}$ and $\hat{\theta}$, which point in the directions of increasing polar coordinates r and θ, *do not* satisfy the above conditions and hence are *not* a coordinate basis.

(c) Show that if instead the θ basis vector is taken to be $\vec{e}_\theta = r\hat{\theta}$, then the new basis vectors are a coordinate basis.

(d) Hence show that the velocity vector in a polar coordinate basis has components $(v_r, v_\theta/r)$, where v_r and v_θ are the usual speeds along the respective directions.

4.1. Consider a 2D space of constant curvature which can be pictured in 3D as the surface of a table-tennis ball. Would a 2D being confined to this space have any equivalents of our concepts of "inside" and "outside" surfaces?

4.2. Can a 1D space be curved?

4.3. Write out the covariant derivative $T_{\alpha\beta;\gamma}$ in full, where $T_{\alpha\beta}$ is an arbitrary tensor.

4.4. Calculate the Christoffel symbols for the surface of a sphere of radius R using the usual spherical polar angles (θ, ϕ) as coordinates.

4.5. (a) Use the Euler-Lagrange equations to show that the shortest path between two points on a sphere satisfies the relations

$$\begin{aligned} \sin^2(\theta)\, \dot{\phi} &= k, \\ \ddot{\theta} &= \dot{\phi}^2 \sin\theta \cos\theta, \end{aligned}$$

where k is a constant, and θ and ϕ are the usual angles of spherical polar coordinates.

(b) Show that circles of constant longitude satisfy these equations but circles of constant latitude do not, except in one case.

4.6. (a) Starting from the flat spacetime metric

$$ds^2 = c^2\, dT^2 - dX^2 - dY^2 - dZ^2,$$

and applying the coordinate transform

$$\begin{aligned} T &= t, \\ X &= x \cos\omega t - y \sin\omega t, \\ Y &= x \sin\omega t + y \cos\omega t, \\ Z &= z, \end{aligned}$$

show that the metric becomes:

$$ds^2 = \left[c^2 - \omega^2 \left(x^2 + y^2\right)\right] dt^2 + 2\omega y\, dx\, dt - 2\omega x\, dy\, dt - dx^2 - dy^2 - dz^2.$$

(b) Write down the Lagrangian corresponding to this metric.

(c) Apply the Euler-Lagrange equations to this Lagrangian and show that the equations of geodesic motion are:

$$\begin{aligned} \ddot{t} &= 0, \\ \ddot{x} - \omega^2 x \dot{t}^2 - 2\omega \dot{y}\dot{t} &= 0, \\ \ddot{y} - \omega^2 y \dot{t}^2 + 2\omega \dot{x}\dot{t} &= 0, \\ \ddot{z} &= 0, \end{aligned}$$

where the dots represent differentiation with respect to proper time.

(d) Write down the value of the connection coefficients ${\Gamma^x}_{yt}$ and ${\Gamma^y}_{tt}$ from the above equations.

(e) Introducing the three-vectors $\mathbf{r} = (x, y, z)$ and $\boldsymbol{\omega} = (0, 0, \omega)$, show from these equations that

$$\frac{d^2\mathbf{r}}{dt^2} + \boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r} + 2\boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt} = 0.$$

What are the physical meanings of the terms in this expression?

Why have they arisen from the coordinate transformation specified?

4.7. Show that the covariant derivative obeys Leibniz' product rule, e.g. given a tensor $T^{\alpha\beta} = U^\alpha V^\beta$, show that

$${T^{\alpha\beta}}_{;\gamma} = {U^\alpha}_{;\gamma} V^\beta + U^\alpha {V^\beta}_{;\gamma}.$$

4.8. The interval of a static, spherically symmetric spacetime can be written

$$ds^2 = A(r)\, dt^2 - B(r)\, dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2.$$

(a) Use the Euler-Lagrange equations to derive the equations of geodesic motion and hence show that ${\Gamma^t}_{tr} = A'/2A$, ${\Gamma^r}_{tt} = A'/2B$, ${\Gamma^r}_{rr} = B'/2B$, ${\Gamma^r}_{\theta\theta} = -r/B$, ${\Gamma^r}_{\phi\phi} = -(r\sin^2\theta)/B$, ${\Gamma^\theta}_{r\theta} = 1/r$, ${\Gamma^\theta}_{\phi\phi} = -\sin\theta\cos\theta$, ${\Gamma^\phi}_{r\phi} = 1/r$, ${\Gamma^\phi}_{\theta\phi} = \cot\theta$, with all others zero. The dashes here denote differentiation with respect to r

[These coefficients are needed for Schwarzschild's solution. You need not calculate them all, just make sure that you understand how to go about it in principle.]

(b) The "proper" acceleration, a, is the acceleration felt by an observer and can be calculated from the norm of the four-acceleration:

$$a^2 = -\vec{A} \cdot \vec{A} = -g_{\alpha\beta} A^\alpha A^\beta.$$

Use this to show that the proper acceleration felt by an astronaut who uses a rocket to remain stationary in terms of the coordinates (r, θ, ϕ) is given by

$$a^2 = \frac{1}{B} \left(\frac{c^2 A'}{2A} \right)^2.$$

(c) What is the proper acceleration of an astronaut following a geodesic path in this metric?

4.9. * It is possible to have a connection in a space without a metric; in this sense it is a more fundamental concept. For instance, independently of any metric one can derive the transformation properties of the connection in order that the following

$$V^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\gamma\beta} V^\gamma,$$

are the components of a tensor.

(a) Use this approach to show that the connection must transform as follows

$$\Gamma^{\alpha'}{}_{\beta'\gamma'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x^{\beta'}} \frac{\partial x^\gamma}{\partial x^{\gamma'}} \Gamma^\alpha{}_{\beta\gamma} - \frac{\partial x^\beta}{\partial x^{\beta'}} \frac{\partial x^\gamma}{\partial x^{\gamma'}} \frac{\partial^2 x^{\alpha'}}{\partial x^\beta \partial x^\gamma}.$$

(b) Show therefore that

$$T^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\beta\gamma} - \Gamma^\alpha{}_{\gamma\beta},$$

is a tensor, known as the *torsion*. In GR the connection is assumed to be torsionless and therefore symmetric in its lower indices.

4.10. * The connection is not unique to GR and can come up in Newtonian physics applied to curved coordinate systems, although often more elementary means can lead to the same results. Consider for example the continuity equation of fluid mechanics:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,$$

where ρ is the density and $\mathbf{v}$ is the fluid velocity. By expressing the second term covariantly (i.e. in a way that is tensorially correct in curved coordinate systems which reduces to the usual expression in Cartesian coordinates) show that in cylindrical polar coordinates this is given by

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r}(r \rho v_r) + \frac{1}{r} \frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0,$$

where (v_r, v_θ, v_z) are the components of velocity in the usual orthogonal unit basis vectors aligned with the polar coordinates.

[1] Hence show that in an axi-symmetric accretion disc in which v_r is independent of z

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r}(r \Sigma v_r) = 0,$$

where Σ is the vertically-integrated surface density of the disc.

[1] This part is very off-topic so don't worry if you can't manage it.

4.11. * There is no need to distinguish between "up" and "down" indices in Cartesian coordinates and conventionally all indices are down. Thus the Laplacian operator can be written as $\nabla^2\psi = \partial_i\partial_i\psi$ in Cartesian tensor notation.

Work out a fully-covariant form of the Laplacian and hence prove the well-known-but-rarely-proven relation for the Laplacian in spherical polars:

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}.$$

5.1. In lectures it was stated that the covariant derivative of the metric was zero, $\nabla\mathbf{g} = 0$.

(a) Use this to show the often-useful property that index-raising and lowering operations can be moved through covariant differentiation, for example

$$V_{\alpha;\beta} = (g_{\alpha\gamma}V^\gamma)_{;\beta} = g_{\alpha\gamma}V^\gamma{}_{;\beta}$$

(b) Hence, given the relation

$$V_{\alpha;\beta\gamma} - V_{\alpha;\gamma\beta} = R^\rho{}_{\alpha\beta\gamma}V_\rho$$

show that

$$V^\alpha{}_{;\beta\gamma} - V^\alpha{}_{;\gamma\beta} = R_\rho{}^\alpha{}_{\beta\gamma}V^\rho$$

a relation used in lectures when discussing the Riemann tensor.

5.2. Calculation of the Riemann tensor in one of the most tedious in GR, however, it is not difficult – in principle – and is worth doing for the simplest case of all, the 2-sphere of radius a, labelled in terms of the spherical polar angles θ and ϕ for which

$$ds^2 = a^2\,d\theta^2 + a^2\sin^2\theta\,d\phi^2,$$

and the only non-zero connection coefficients are $\Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta$ and $\Gamma^\phi{}_{\phi\theta} = \Gamma^\phi{}_{\theta\phi} = \cot\theta$ (see problem sheet 4).

Evaluate the Riemann and Ricci tensors, and thus show that the Ricci scalar $R = -2/a^2$.

5.3. When developing a model of the Universe, Einstein added an extra term to the field equations so that they read

$$R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta} + \Lambda g^{\alpha\beta} = kT^{\alpha\beta}$$

where Λ is the "cosmological constant" and $k = -8\pi G/c^4$.

(a) Prove that these equations still satisfy the condition $T^{\alpha\beta}{}_{;\alpha} = 0$.

(b) Show that the Ricci scalar $R = -kT + 4\Lambda$ where, as in lectures, $T = g_{\alpha\beta}T^{\alpha\beta}$.

(c) Hence show that

$$R^{\alpha\beta} = k\left(T^{\alpha\beta} - \frac{1}{2}Tg^{\alpha\beta}\right) + \Lambda g^{\alpha\beta}$$

(d) Therefore, by modifying the Newtonian limit calculation used in lectures to derive the value of k, show that in the limit of slow motion and weak fields, the 00 component of the field equations becomes

$$\nabla^2 \phi = 4\pi G \rho - \Lambda c^2.$$

(e) Show then that the Newtonian potential at distance r from a spherically symmetric mass M is

$$\phi = -\frac{GM}{r} - \frac{\Lambda c^2 r^2}{6}.$$

(f) What would be the physical effect of the cosmological constant term?

What effect would it have upon the orbital periods of the planets?

(g) In 1997 evidence was found in observations of supernovae that indicates that the cosmological constant is not zero, but has a value of $\Lambda = 1.2 \times 10^{-52}\,\mathrm{m}^{-2}$. Estimate whether this would have observable effects upon the orbits of the planets.

5.4. Show that, in the absence of a cosmological constant, the field equations in free-space can be written as:

$$R^{\alpha\beta} = 0.$$

Does this imply that free-space must be flat?

5.5. * Starting from the connection coefficients of part (a) of Q**4.8**, show that the non-zero coefficients of the Ricci tensor for a spherically symmetric spacetime are:

$$\begin{aligned}
R_{tt} &= -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB}, \\
R_{rr} &= \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB}, \\
R_{\theta\theta} &= \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right), \\
R_{\phi\phi} &= R_{\theta\theta} \sin^2\theta,
\end{aligned}$$

where the dashes and double-dashes indicates first and second derivatives with respect to r.

Set equal to zero (see previous question), these are the equations that lead to the Schwarzschild metric with $A(r) = c^2(1 - 2GM/c^2 r)$, $B(r) = (1 - 2GM/c^2 r)^{-1}$.

5.6. A line parameterised by λ and with tangent vector $\vec{U}$ obeys the relation

$$\nabla_{\vec{U}} \vec{U} = f \vec{U},$$

where $f = f(\lambda)$ is a function of λ.

(a) Show that, by parameterising the line in terms of a new parameter μ such that

$$\frac{d^2\mu}{d\mu d\lambda} = f,$$

the line is a geodesic.

(b) Hence show that if f is constant then

$$\mu = Ae^{f\lambda} + B,$$

where A and B are constants.

5.7. Show that the worldline which maximises the integral $\int L\, d\lambda$, where $L = g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta$, and dots denote derivatives with respect to λ, satisfies the relations

$$\ddot{x}^\alpha + \Gamma^\alpha{}_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = 0,$$

where the connection is given by the Levi-Civita relation.

5.8. * *Linearised GR:* when gravitational fields are weak, one can find coordinates throughout spacetime for which $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ with $|h_{\alpha\beta}| \ll 1$.

(a) Show that to first order in h, the Ricci tensor can be written as

$$R_{\alpha\beta} = \frac{1}{2}\eta^{\gamma\delta}\left(h_{\delta\gamma,\alpha\beta} + h_{\alpha\beta,\delta\gamma} - h_{\alpha\gamma,\delta\beta} - h_{\delta\beta,\alpha\gamma}\right).$$

(b) Hence show that to first-order in h, the Ricci scalar is given by

$$R = \Box h - h^{\alpha\beta}{}_{,\alpha\beta}\,,$$

where h is the trace, i,e $h = \eta^{\alpha\beta}h_{\alpha\beta}$, $\Box$ is the d'Alembertian

$$\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2,$$

and index raising and lowering involves η rather than g.

(c) Finally, show that the field equations can be written

$$h_{,\alpha\beta} + \Box h_{\alpha\beta} - \eta^{\gamma\delta}\left(h_{\alpha\gamma,\delta\beta} + h_{\delta\beta,\alpha\gamma}\right) - \left(\Box h - h^{\sigma\rho}{}_{,\sigma\rho}\right)\eta_{\alpha\beta} = -\frac{16\pi G}{c^4}T_{\alpha\beta}.$$

(This equation, which with a careful choice of coordinates can be greatly simplified, is the starting point for the theory of gravitational waves.)

5.9. By working in inertial coordinates, prove the *Bianchi identity*, given, but not proved, in the handouts:

$$R_{\alpha\beta\gamma\delta;\mu} + R_{\alpha\beta\mu\gamma;\delta} + R_{\alpha\beta\delta\mu;\gamma} = 0.$$

[Note the way the last three indices are cyclicly permuted.]

5.10. Prove that the Ricci tensor is symmetric.

5.11. Show that in one-dimension there is no Riemann tensor while in two dimensions it has only one independent component.

5.12. * Show that the number of independent components of the Riemann tensor in N dimensions is given by

$$\frac{1}{12}N^2(N^2-1).$$

6.1. The non-zero coefficients of Einstein's field equations in empty space for a spherically symmetric metric of the form

$$ds^2 = A(r)\,dt^2 - B(r)\,dr^2 - r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right),$$

are as follows:

$$R_{tt} = -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB} = 0, \tag{5}$$

$$R_{rr} = \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB} = 0, \tag{6}$$

$$R_{\theta\theta} = \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) = 0. \tag{7}$$

(The $R_{\phi\phi}$ component carries the same information as the $R_{\theta\theta}$ component.)

(a) By adding $B\times$ Eq. 5 to $A\times$ Eq. 6 show that $AB = \alpha$, a constant.

(b) Hence use Eq. 7 to show that $d(rA)/dr = \alpha$, and so

$$A(r) = \alpha\left(1 + \frac{k}{r}\right),$$

where k is another integration constant.

(c) Finally, by considering the weak field limit, justify the Schwarzschild metric.

6.2. With the cosmological constant, the field equations in empty-space become

$$R^{\alpha\beta} = \Lambda g^{\alpha\beta}.$$

Write down the modified versions of Eqs 5, 6 and 7 from Q**6.1**, and repeat the working to show that the metric becomes

$$ds^2 = c^2\left(1 - \frac{2GM}{c^2 r} - \frac{\Lambda r^2}{3}\right)dt^2 - \left(1 - \frac{2GM}{c^2 r} - \frac{\Lambda r^2}{3}\right)^{-1}dr^2 - r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right).$$

6.3. Person A, stationary at $r = r_A$ from a mass M (Schwarzschild radial coordinates), regularly sends pulses of light in the radial direction to person B who is stationary at $r = r_B > r_A$.

(a) Show that along the path of the light-pulse

$$c\,dt = \frac{dr}{1 - 2GM/c^2 r}.$$

(b) Show that each pulse take the same coordinate time to travel from A to B.

(c) Hence show that, if A transmits pulses at rate ν_A, then B receives them at rate ν_B where

$$\frac{\nu_B}{\nu_A} = \left(\frac{1 - 2GM/c^2 r_A}{1 - 2GM/c^2 r_B}\right)^{1/2}.$$

(d) What is the physical meaning of the equation of the previous part?

(e) By how much would the readings of two clocks differ after one year, if one was on the surface of Earth while the other was far from the Earth but stationary with respect to it, assuming that they were initially synchronised?

(f) The spectrum of the surface of a white dwarf of radius $R = 6000\,\mathrm{km}$, mass $M = 1\,\mathrm{M}_\odot$ shows strong absorption at the wavelength of Hα, which has laboratory rest wavelength $\lambda = 656.276\,\mathrm{nm}$.

i. What wavelength would be measured on Earth, assuming that the white dwarf is stationary with respect to Earth?
[You may ignore any gravitational effects due to Earth.]

ii. An astronomer takes a spectrum of this white dwarf and mis-interprets the offset wavelength as a Doppler shift. What spurious velocity along the line-of-sight would the astronomer measure?

(g) Which has a more significant effect upon the clocks in a typical commercial air flight ($v \approx 900\,\mathrm{km\,hr}^{-1}$, $h \approx 10{,}000\,\mathrm{m}$), the speeding up of time from being at a higher gravitational potential than the ground or the slowing down from special relativistic time dilation?

6.4. Obtain an expression for the coordinate time taken for light to travel radially from r_1 to r_2 in Schwarzschild coordinates.

Calculate the coordinate time taken for light to travel from Earth to Mercury (0.38 AU from the Sun) and back to Earth, assuming a purely radial path, and compare with the simple formula $\Delta t = 2\Delta r/c$.

6.5. An observer is stationary at Schwarzschild radial coordinate r from a mass M. Starting from

$$A^\alpha = \frac{DU^\alpha}{D\tau} = \frac{dU^\alpha}{d\tau} + \Gamma^\alpha{}_{\beta\gamma} U^\beta U^\gamma,$$

show that the observer experiences an acceleration a given by

$$a = \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} \frac{GM}{r^2}.$$

In which direction does this acceleration point?

6.6. In empty space $T^{\alpha\beta} = 0$, and the field equations reduce to $R^{\alpha\beta} = 0$. A tensor that is zero in one frame is zero in all frames. Thus there is no curvature in empty space. True or false?

6.7. No general method has been established for proving whether a given metric has a singularity. A guide is to calculate scalar invariants.

Look up the coefficients of the Riemann tensor for the Schwarzschild geometry and hence calculate the value of the scalar

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}.$$

Show that it is finite at the Schwarzschild radius but singular at $r = 0$.

Why is the Ricci scalar not a useful guide in this case?

6.8. How large would a sphere of material with the same density as air have to be for its Schwarzschild radius to exceed its own radius? Compare the radius you estimate with the size of the solar system.

7.1. A particle is dropped from rest at a radius $r = r_0$ in Schwarzschild coordinates from a black-hole of mass M.

(a) Show that at radius r

$$\frac{dr}{d\tau} = -\sqrt{\frac{2GM}{r} - \frac{2GM}{r_0}},$$

where τ is the particle's proper time.

(b) Hence show that the total proper time for the particle to reach the singularity at $r = 0$ is given by

$$\tau = \pi\sqrt{\frac{r_0^3}{8GM}}.$$

(c) Calculate the proper time taken for a particle dropped from the event horizon of the black-hole at the centre of our Galaxy ($M = 4.5 \times 10^6\,\mathrm{M}_\odot$) to reach the singularity.

7.2. An observer stationary at radius $r > R_S$ (Schwarzschild coordinates) measures the speed of the particle of Q**7.1** as it passes by having started at $r_0 > r$. Show the following:

(a) The proper "ruler" distance dl measured by the observer corresponding to a change in radial coordinate dr is

$$dl = \frac{dr}{(1 - 2\mu/r)^{1/2}},$$

where $\mu = GM/c^2$.

(b) The derivative of coordinate time t with respect to the particle's proper time τ_p is given by

$$\left(\frac{dt}{d\tau_p}\right)^2 = \dot{t}^2 = \left(1 - \frac{2\mu}{r}\right)^{-1} + \left(1 - \frac{2\mu}{r}\right)^{-2}\left(\frac{\dot{r}}{c}\right)^2.$$

(c) The derivative of the observer's proper time τ_o with respect to coordinate time t is given by

$$\frac{d\tau_o}{dt} = \left(1 - \frac{2\mu}{r}\right)^{1/2}.$$

(d) Thus, combining the above results, show that if the observer is very close to the event horizon, the particle will pass by at the speed of light, independent of its initial radius.

7.3. A particle is set in an orbit around a black-hole of mass M starting from radius $r_0 \gg \mu = GM/c^2$ with a purely angular motion at speed $v = r\dot{\phi} = h/r$. The speed v is much less than the circular orbital speed at r_0 so that as it orbits the particle passes close to the black-hole.

(a) Using the energy equation from lectures, show that $|k^2 - 1| \ll 1$.

(b) Hence, by considering the effective potential, show that the particle will be captured by the black-hole if

$$v < 4\left(\frac{\mu}{r_0}\right)c.$$

(c) Calculate the maximum value of v for $M = 1\,\mathrm{M}_\odot$, and $r_0 = 1\,\mathrm{AU}$.

7.4. Show that the radius of the inner maximum of the Schwarzschild potential – provided that there is such a maximum – is minimised as $h \to \infty$.

Hence show that the closest one can pass by a black-hole of mass M without being captured is given by $r = 3GM/c^2$.

7.5. If the Sun was replaced by a black-hole of the same mass, how accurately would a laser from Earth have to be directed to ensure that its beam was captured? [Ignore Earth's motion.]

7.6. A satellite of mass m orbits an object of mass M at constant distance r.

(a) Writing as usual $\mu = GM/c^2$, show that the energy constant k is given by

$$k^2 = 1 - \left(\frac{r - 4\mu}{r - 3\mu}\right)\frac{\mu}{r}.$$

(b) Hence show that at $r \gg \mu$ the energy of the particle is given by

$$E \approx mc^2 - \frac{GMm}{2r}.$$

Give a physical interpretation for this result.

(c) Obtain an expression for the ratio α of the rate at which time passes for the particle compared to the rate it passes for an observer stationary at infinity.

Calculate α for the last stable circular orbit.

Show that for $r \gg \mu$

$$\alpha \approx 1 - \frac{3GM}{2c^2 r}.$$

Give an interpretation of this result in terms of gravitational and special-relativistic time dilation factors.

(d) The GPS satellites provide a famous practical example of GR. The GPS satellites orbit at a radius of 26,600 km around Earth which has mass 5.98×10^{24} kg and radius 6370 km.

Calculate the rate at which the GPS satellites gain or lose compared to clocks on Earth, and hence, given that they provide positional information by timing, calculate the positional error that could result after one day of ignoring relativistic effects.

7.7. An alternative (but deeply unattractive) explanation for the 43"/century "anomalous" precession of the perihelion of Mercury is that Newton's law of gravity should be modified from $1/r^2$ to $1/r^{2+\epsilon}$ where ϵ is small but non-zero.

Calculate the value of ϵ that matches Mercury's precession rate.

7.8. Both special relativity and general relativity provide the opportunity to time-travel into the future. However, to do so with SR alone requires an enormous expenditure of energy in order to accelerate near to the speed of light and then to slow down.

(a) By considering the form of the effective potential of massive particles, show that it is possible to use orbital motion around a Schwarzschild black-hole to time-travel "on the cheap" with little or no expenditure of energy.

(b) Starting from a large radius, what value of h in units of μc would allow the most effective time-travel for minimal expenditure of energy.

(c) By what factor could one step forward into the future by these means?

7.9. * The equation of geodesic deviation was quoted in lectures as

$$\frac{D^2 w^\alpha}{D\lambda^2} + R^\alpha{}_{\gamma\beta\delta}\,\dot{x}^\gamma \dot{x}^\delta w^\beta = 0,$$

where $\vec{W}$ is a vector separating two particles in free-fall. The first terms is the total derivative which allows for arbitrary coordinates. If we can stick to Cartesians, it reduces to $d^2 w^\alpha/d\lambda^2$.

Consider two massive particles falling vertically down along a radial line towards the North pole on Earth, separated vertically by a distance s. Assuming non-relativistic motion, and defining the z-axis to be vertical show that

$$\frac{ds^2}{dt^2} = -c^2 R^z{}_{tzt}\, s,$$

i.e. the two particles accelerate relative to each other by an amount proportional to their separation, providing a way to measure one of the Riemann tensor components.

7.10. Calculate the minimum angle above the surface of a neutron star of mass $M = 2.5\,\mathrm{M}_\odot$, and radius $R = 8\,\mathrm{km}$ at which one would have to direct a laser beam in order for the light to escape the star.

[Warning: you may be tempted to write down that

$$\tan\alpha = \frac{1}{r}\frac{dr}{d\phi},$$

where α is the angle to the surface, but this is not quite right: think in terms of the proper distance moved given small changes in coordinates r and ϕ.]

7.11. * Consider orbits of particle subject to a central attractive force of the form $F \propto r^\alpha$.

(a) Derive a condition on α such that a whole number N of radial "epicycles" are completed within one orbit. (The orbit is then closed.)

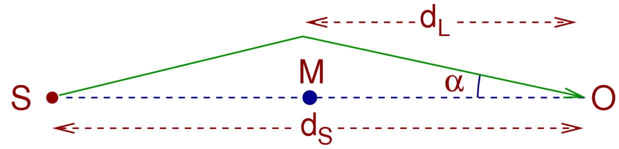
(b) Comment on the values of α for $N = 1$ and $N = 2$.

(c) For what values of α are circular orbits stable?

(d) What is the precession angle per orbit for $\alpha = -1$ and is the precession in this case prograde or retrograde?

7.12. Consider the photon energy equation for $r < 2\mu$. Show that in this case $\dot{r}$ can never be zero and so photons can only travel towards smaller or larger r but never switch direction.

8.1. When a mass M lies along the line of sight to a source S, it bends the light to form an "Einstein ring". The figure below shows the geometry of the ring formation.



(a) Use the light-deflection formula from lectures to show that the angular radius α is given by

$$\alpha^2 = \frac{4GM}{c^2}\left(\frac{1}{d_L} - \frac{1}{d_S}\right).$$

(b) Calculate the angular radius of the Einstein ring formed when $M = 1\,\mathrm{M}_\odot$, $d_S = 8\,\mathrm{kpc}$ and $d_L = 4\,\mathrm{kpc}$.

8.2. Light can orbit in a circle at radius $r = 3GM/c^2$ from a mass M, but such orbits are unstable. By approximating the radial '"energy" equation for photons for a small perturbation from the exact circular orbit radius, show that the perturbation grows as e^{ϕ} where ϕ is the azimuthal angle travelled by the photons.

8.3. Calculate the coordinate time taken for light to complete one circular orbit of a $10\,\mathrm{M}_\odot$ black-hole.

8.4. Were you to fall into a black-hole, would you see the singularity at $r = 0$ once you crossed the event horizon? Justify your answer.

8.5. Obtain expressions for the distance between two points on the same radial line at radii r_1 and r_2 (Schwarzschild coordinates) from a mass M in two ways:

(a) By evaluating the proper or "ruler" distance,

(b) By taking c times the light travel time as measured by an observer stationary at $r = r_2$.

Hence calculate the difference between these two distances and the coordinate distance $r_2 - r_1$ for a radial line from the surface of the Sun to Earth.

[Ignore any motion of either the Sun or Earth.]

8.6. In terms of Schwarzschild coordinates r and t, Kruskal's coordinates u and v are given for $r > 2\mu$ by

$$\begin{aligned} v &= \left(\frac{r}{2\mu} - 1\right)^{1/2} \exp\left(\frac{r}{4\mu}\right) \sinh\left(\frac{ct}{4\mu}\right), \\ u &= \left(\frac{r}{2\mu} - 1\right)^{1/2} \exp\left(\frac{r}{4\mu}\right) \cosh\left(\frac{ct}{4\mu}\right), \end{aligned}$$

whereas for $r < 2\mu$

$$\begin{aligned} v &= \left(1 - \frac{r}{2\mu}\right)^{1/2} \exp\left(\frac{r}{4\mu}\right) \cosh\left(\frac{ct}{4\mu}\right), \\ u &= \left(1 - \frac{r}{2\mu}\right)^{1/2} \exp\left(\frac{r}{4\mu}\right) \sinh\left(\frac{ct}{4\mu}\right), \end{aligned}$$

Show that

(a) lines of constant r are hyperbolae in Kruskal coordinates,

(b) lines of constant t are straight lines through the origin, and, in particular, that straight lines running through the origin at $\pm 45^\circ$ represent $t = \pm\infty$, $r = 2\mu$.

(c) the interval is given by

$$ds^2 = \frac{32\mu^3}{r} \exp\left(-\frac{r}{2\mu}\right) \left(dv^2 - du^2\right) - r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right),$$

(d) u is spacelike and v is timelike, for all r

(e) photons on radial paths travel on the same $\pm 45^\circ$ lines in u, v coordinates that they do in Minkowski spacetime diagrams

(f) there is one line of $r = 0$ in Kruskal space that can only ever be in your future and another that can only ever be in your past.

8.7. During an experiment near a black-hole, Alice's jet-pack fails. Her selfish fellow astronaut Bob's main concern is for his own peace of mind in the event of his seeing Alice tested to breaking point by tidal forces.

Use a Kruskal diagram to show graphically that, however long he waits, Bob will never see Alice cross the event horizon. (Thus as long as the black-hole is massive enough to swallow Alice in one piece, Bob need not worry about future sleepless nights.)

In a fit of remorse, Bob decides that he will at least keep sending Alice signals while he can still see her, although this threatens to be forever as she appears to him to be stuck at the event horizon. Show again from the Kruskal diagram that there will come a time when Bob should stop sending signals to Alice because she will in fact have reached the singularity.

[You may assume that Alice and Bob are on the same radial line.]

8.8. "Advanced Eddington-Finkelstein" coordinates use the constant of integration in the equation for worldlines of radially infalling photons to replace the coordinate t in the Schwarzschild metric with t' given by

$$ct' = ct + 2\mu \ln \left| \frac{r}{2\mu} - 1 \right| .$$

(a) Obtain an expression for the interval in the Schwarzschild geometry using t' and r rather than t and r (you may ignore the angular terms).

(b) Derive relations for t' as a function of r for in- and out-going photons and use these to plot the light-cone structure near a black-hole in these coordinates.

(c) Use this structure to show that once inside a black-hole, there is no escape.

9.1. Derive the fluid equation starting from the acceleration and Friedmann equations.

9.2. Integrate the Friedmann equation for a flat, matter-only universe (Einstein-de Sitter model) to show that $R \propto t^{2/3}$.

Hence show that the age of such a universe is given by

$$t_0 = \frac{2}{3} t_H,$$

where $t_H = H_0^{-1}$ is the "Hubble time" and $H_0 = H(t_0)$ is Hubble's constant at time t_0.

9.3. Find the corresponding results to the previous question for a flat, radiation-only universe for which $\rho_R \propto R^{-4}$. (This describes the early universe.)

9.4. Use the fluid equation to show that a fluid for which $p = -\rho c^2$ does not change in density as the universe expands. Comment on this result.

9.5. Show that, in the Einstein-de Sitter universe, an object of fixed proper length seen at different redshifts subtends a minimum angle at one particular redshift, and calculate the value of this redshift.

Estimate the minimum angular diameter of a galaxy of diameter 20 kpc assuming that our Universe follows the Einstein-de Sitter model with $H_0 = 72\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

9.6. The different components of the Universe (matter, radiation and the cosmological constant) all have equations of state of the form $p = w\rho c^2$.

(a) Write down the values of w for each component.

(b) Use the acceleration equation to obtain a condition on w for a component if it is to accelerate the rate of expansion of the Universe. (The density is always assumed to be positive.)

9.7. The "surface brightness" of an object is the flux from it measured at Earth per unit solid angle or equivalently square arcsecond on the sky. Show that the surface brightness of the same object seen at different redshifts scales as $(1 + z)^{-4}$.

9.8. Calculate the radius of a sphere of density equal to the critical density that contains a mass equal to that of the Sun, assuming that $H_0 = 72\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

9.9. (a) Show that for a universe governed by matter and the cosmological constant alone, Friedmann's equation can be written as

$$H(z) = H_0 \left(\Omega_M(1+z)^3 + \Omega_\Lambda - (\Omega_M + \Omega_\Lambda - 1)(1+z)^2\right)^{1/2}.$$

where H_0 is the present day value of Hubble's constant, $H(z)$ is Hubble's constant as a function of redshift z and Ω_M and Ω_Λ are the present day ratios of the matter and cosmological constant densities to the critical density $\rho_C = 3H_0^2/8\pi G$.

(b) Which term in the above relation represents curvature?

(c) When discussing distances, the following integral was needed to evaluate the comoving radial coordinate χ:

$$\chi = \int_0^z \frac{c\,dz}{H(z)}.$$

Show from the relation of part (a) that this integral is a monotonically increasing function of Ω_Λ.

(d) What is the relation of the result of part (c) to the use of supernovae in cosmology?

9.10. (a) Starting from Einstein's field equations in the contravariant form

$$R^{\alpha\beta} = -k\left(T^{\alpha\beta} - \frac{1}{2}T g^{\alpha\beta}\right) + \Lambda g^{\alpha\beta},$$

and given that for the FRW metric

$$R^{tt} = \frac{3}{c^4}\frac{\ddot{R}}{R},$$

derive the following form of the acceleration equation

$$\ddot{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)R + \frac{1}{3}\Lambda c^2 R.$$

where ρ does *not* include a cosmological constant component.

(b) Show that this is equivalent to the equation derived in lectures if one adopts the view that the cosmological constant is a fluid of density

$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi G},$$

and pressure $p_\Lambda = -\rho_\Lambda c^2$.

9.11. There are no temperature gradients in a homogeneous Universe and so no heat transfer and thus one can write

$$dU + p\,dV = 0,$$

where U is the internal energy of a volume V of the Universe and p is the pressure. Use this and the relativistic mass-energy relation to prove the fluid equation:

$$\dot{\rho} + 3H(t)\left(\rho + \frac{p}{c^2}\right) = 0,$$

where ρ is the density, p is the pressure, $H(t)$ is Hubble's "constant" as a function of time and the dot denotes a derivative with respect to cosmic time t.

9.12. "CODEX" is a proposed high-resolution spectrograph designed to measure the rate of change of the redshift $\dot{z}$ of distant objects as a way to measure past values of Hubble's constant. This is in principle visible as a change in the apparent recession velocity of the objects.

(a) Show that

$$\dot{z} = (1+z)H_0 - H(z),$$

where H_0 is the present day value of Hubble's constant and $H(z)$ is the value of Hubble's constant at the time light from objects at redshift z was emitted.

(b) Hence calculate the rate of change of recession velocity of an object of redshift $z = 1$, assuming an Einstein-de Sitter universe with $R(t) \propto t^{2/3}$.
[Present day value of Hubble's constant $H_0 = 72\,\mathrm{km\,s^{-1}Mpc^{-1}}$.]

9.13. A ballistic projectile is fired from the origin in an Einstein-de Sitter universe ($R(t) \propto t^{2/3}$) towards a galaxy nearby enough to have a non-relativistic recession speed due to universal expansion.

(a) Show that

$$\ddot{\chi} = -2\frac{\dot{R}}{R}\dot{\chi},$$

where the dots denote derivatives with respect to the proper time τ of the projectile.

(b) * Hence show that in order for the projectile to reach the galaxy it must be fired at a speed greater than half the apparent recession speed of the galaxy as measured at the time of firing.

10.1. The generalised coordinates of GR can be confusing; the TT gauge of gravitational waves is a good example of this. This question illustrates this. A weak gravitational field is described by $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ where $|h_{\alpha\beta}| \ll 1$.

(a) Use the Levi-Civita equation (handout 3) to show that to first order the connection can be written

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2}\eta^{\alpha\sigma}\left(h_{\beta\sigma,\gamma} + h_{\sigma\gamma,\beta} - h_{\beta\gamma,\sigma}\right).$$

(b) In the TT gauge, $h_{\alpha 0} = h_{0\alpha} = 0$. Use the general equations of motion

$$\ddot{x}^{\alpha} + \Gamma^{\alpha}{}_{\beta\gamma}\dot{x}^{\beta}\dot{x}^{\gamma} = 0,$$

and the result of part (a) to show that freely-floating particles which are initially stationary ($\dot{x}^i = 0$, $i = 1, 2, 3$) will remain stationary in the presence of gravitational waves.

(c) The previous result is misleading: the particles are stationary in *coordinates*, but the physically measurable distance between particles is variable, i.e. the TT coordinates track the particles. To see this show, from the form of $h^{\alpha\beta}$ derived in lectures that the distance l from the origin of a free particle at TT coordinates (x, y) as a gravitational wave of angular frequency Ω passes in the z direction is given by $l^2 = x^i A_{ij} x^j$, where i and j are only summed over x and y and the 2x2 matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 1 - a\cos\Omega t & -b\cos\Omega t \\ -b\cos\Omega t & 1 + a\cos\Omega t \end{pmatrix}.$$

(In general the time-dependent terms could also include phase shifts.)

(d) Use the relation of part (c) to justify the standard elliptical distortion patterns of gravitational waves when they pass through a set of particles initially arranged in a circle.

10.2. Using the formula from lectures for the amplitude of gravitational waves produced by a variable mass quadrupole at distance r

$$\bar{h}^{ij} = -\frac{2G}{c^4 r}\frac{d^2 I^{ij}}{dt^2},$$

where

$$I^{ij} = \int \rho x^i x^j \, dV,$$

discuss the feasibility of setting up a calibration source for the current laser interferometers consisting of two equal masses rotated around a vertical axis. Assume a sensitivity of $h \sim 10^{-21}$ at frequencies $\sim 100\,\mathrm{Hz}$.

10.3. Two particles of equal mass M separated by a moving in circular orbits around their centre of mass lose energy due to the emission of gravitational waves at a rate

$$\frac{dE}{dt} = -\frac{64 G^4 M^5}{5 c^5 a^5}.$$

The energy comes from the shrinkage of the orbit of the two particles.

(a) Using Newtonian mechanics show that the rate of change of the orbital separation is given by

$$\dot{a} = -\frac{128 G^3 M^3}{5 c^5 a^3}.$$

(b) Hence obtain an expression for the time taken for the two particles to spiral together.

(c) Calculate the time for two neutron stars, each with $M = 1.4\,\mathrm{M}_\odot$ to merge starting from an orbital period of 2 hr. [Assume circular orbits throughout.]

(d) The LIGO interferometers are expected to pick up the final in-spiral of pairs of neutron stars when they have reached a gravitational wave frequency of about 50 Hz; seismic noise makes detection difficult at lower frequencies.

Estimate how long a neutron star merger event will last for LIGO.

Would a pair of merging $10\,\mathrm{M}_\odot$ black-holes be detectable over a longer or shorter interval of time?

(e) Qualitatively describe the nature of the gravitational wave signal that will characterise the first part of such mergers, while the two stars can be treated as point masses.

10.4. Use the quadrupole formula to show that a spherically symmetric mass distribution produces no gravitational waves.

10.5. In lectures (see also problem sheet 5) it was stated that the linearised field equations reduce to

$$h_{,\alpha\beta} + \Box h_{\alpha\beta} - \eta^{\gamma\delta}\left(h_{\alpha\gamma,\delta\beta} + h_{\delta\beta,\alpha\gamma}\right) - \left(\Box h - h^{\sigma\rho}{}_{,\sigma\rho}\right)\eta_{\alpha\beta} = 2kT_{\alpha\beta},$$

where $\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$.

Show by applying the Lorenz gauge condition:

$$h^{\alpha\beta}{}_{,\beta} = \frac{1}{2}\eta^{\alpha\beta}h_{,\beta},$$

that the field equations reduce to

$$\Box h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\Box h = 2kT_{\alpha\beta}.$$

10.6. * This question concerns the equivalent in GR of magnetic fields, something that does not come up in Newtonian gravity.

(a) Consider a stress-energy tensor $T^{\alpha\beta}$ that is time-independent ($T^{\alpha\beta}{}_{,0} = 0$, often called "stationary" although there can still be motion, e.g. rotation of a sphere).

In this case, justify why the general solution of the linearised field equations

$$\Box\bar{h}^{\alpha\beta} = 2kT^{\alpha\beta},$$

can be written as

$$\bar{h}^{\alpha\beta}(\mathbf{x}) = \frac{k}{2\pi}\int\frac{T^{\alpha\beta}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,d^3y,$$

where $\mathbf{x}$ is the 3-vector position at which we want the value of $\bar{h}^{\alpha\beta}$, and $\mathbf{y}$ defines the volume element while $k = -8\pi G/c^4$.

(b) For non-relativistic sources, the energy-momentum tensor components are easily seen to be given by

$$T^{00} = \rho c^2, \quad T^{0i} = \rho c u^i, \quad T^{ij} = \rho u^i u^j,$$

where u^i are the components of the 3-velocity. Show from these that to first order in the velocities,

$$\bar{h}^{00} = \frac{4\phi}{c^2}, \quad \bar{h}^{0i} = \frac{A^i}{c}, \quad \bar{h}^{ij} = 0,$$

where

$$\begin{aligned}\phi(\mathbf{x}) &= -G\int\frac{\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,d^3y,\\ A^i(\mathbf{x}) &= -\frac{4G}{c^2}\int\frac{\rho(\mathbf{y})u^i(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,d^3y.\end{aligned}$$

(c) Hence show that

$$h_{00} = h_{11} = h_{22} = h_{33} = \frac{2\phi}{c^2}, \quad h_{0i} = \frac{A_i}{c}.$$

(d) Finally, show that the corresponding approximate line element can be written as

$$ds^2 = c^2 \left(1 + \frac{2\phi}{c^2}\right) dt^2 + 2A_i \, dt \, dx^i - \left(1 - \frac{2\phi}{c^2}\right) dx^i dx^i,$$

with implied summation over the i-index in the final term.

The term in A_i shows that in GR, *motion* of the gravitating mass can affect the line element and hence the orbital motion of nearby test particles. This crops up in the *Kerr metric* for rotating black-holes which unfortunately we do not have the time to cover. ϕ and A_i are analogous to the scalar and vector potentials of electromagnetism.