Notes for PX436, General Relativity

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Foreword:

These notes mostly show the essentials of the lectures, i.e. what I write on the board. The exception to the rule is when I write *pieces of text like this* (outside of the examples). These represent information that I may have said but not written during lectures. I use them when I think it would help you follow the notes.

The notes are very terse, and brief to the point of grammatical inaccuracy. This is because they are notes and are not intended to replace books. I make them available in case you had to miss a lecture or find it difficult to make notes during lectures, but if you rely on these notes only and do not read books, you will struggle.

Lecture 1

Introduction to GR

Objectives:

- *Presentation of some of the background to GR*

Reading: *Rindler chapter 1, Weinberg chapter 1, Foster & Nightingale introduction.*

1.1 Introduction

Newtonian gravity is clearly inconsistent with Special Relativity (SR). Consider Poisson's equation for the gravitational potential ϕ

$$\nabla^2 \phi = 4\pi G \rho,$$

ρ = density. No time derivative $\implies$ gravity instantaneous, and ρ not a Lorentz-invariant.

1.2 What makes gravity special?

Same problems apply to $\nabla^2 \phi = -\rho/\epsilon_0$ from electrostatics, but full Maxwell's equations are Lorentz-invariant.

Something odd about gravity. Consider:

$$\mathbf{F} = m_I \mathbf{a},$$

for the force acting on a mass accelerating at rate $\mathbf{a}$ and

$$\mathbf{F} = m_G \mathbf{g},$$

for the force acting on the same mass in a gravitational field **g**.

This is why we can talk about the acceleration due to gravity

Why is $m_I = m_G$? In Newton's theory this is a remarkable coincidence.

1.2.1 How remarkable?

Galileo, Newton: m_I/m_G same to 1 part in 10^3 (pendulum experiments)

Eötvös (1889): m_I/m_G same to 1 part in 10^9

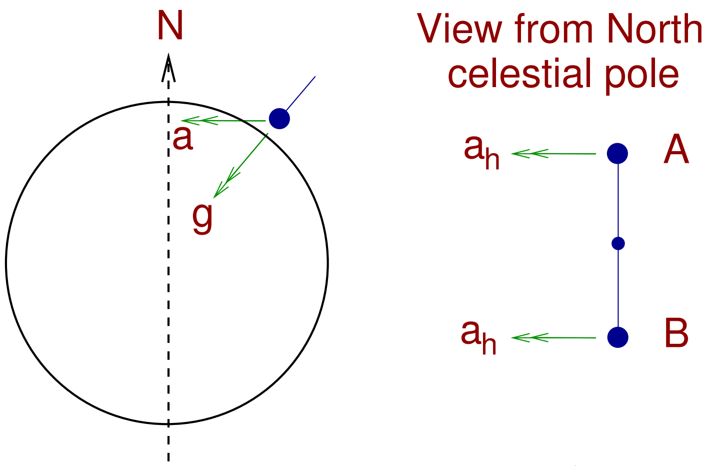

Figure: Eötvös's experiment. Two masses A and B are in balance on a beam suspended by a torsion fibre. If they have a different ratio of inertial and gravitational mass, the horizontal component of centripetal acceleration due to Earth's rotation will cause a torque. None could be measured.

If two masses gravitationally balance, but m_I/m_G differs, there will be a torque on the fibre due to the centripetal acceleration from Earth's rotation.

Dicke et al (1960s): m_I/m_G varies by < 1 part in 10^{12}

1.3 Inertial frames

Definition: in the absence of forces, particles move with constant velocity in inertial frames (straight, at constant speed).

In EM neutral particles can be used to spot an inertial frame, but there are no "neutral" particles in gravity. Are there inertial frames in a gravitational field, even in thought experiments?

What defines "inertial frames" (as important in Special Relativity as in Newtonian gravity)?

Newton: water in a bucket at the North Pole has a curved surface because it rotates relative to the "fixed stars" – Earth not an inertial frame.

Ernst Mach (1893): what if there were no "fixed stars"? Thought that Earth would define its own "inertial frame" – "Mach's Principle" – water surface would be flat. Real physical consequences. e.g. expect acceleration in direction of rotation near massive rotating object, "dragging of inertial frames". No quantitative content however.

Does the weather on Earth require the rest of the Universe?

1.4 Principle of Equivalence

Einstein "explained" $m_I = m_G$ with his principle of equivalence:

> The physics in a freely-falling small laboratory is that of special relativity (SR).

Equivalently, one cannot tell whether a laboratory on Earth is not actually in a rocket accelerating at $1\,g$.

Has real physical content:

e.g. Preditcs that light moves in a straight line at $v = c$ in a freely-falling laboratory. It is a "locally inertial" frame and gravity disappears.

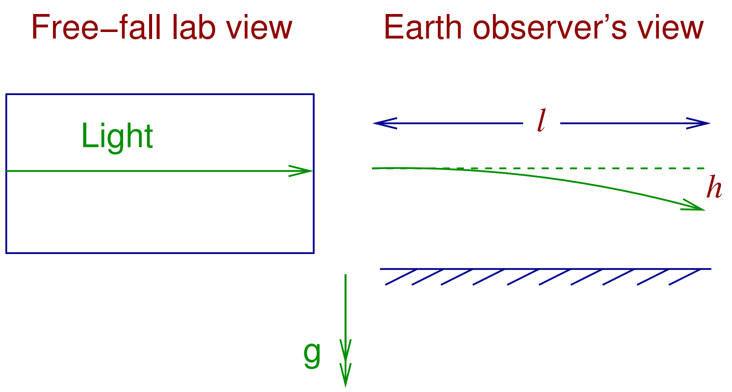

Figure: Light sent across a freely-falling laboratory on the right appears straight, but must appear to bend according to an Earth-based observer since the laboratory accelerates downwards as the light travels across it.

The light takes time

$$t = \frac{l}{c}$$

to cross the lab. Therefore

$$h = \frac{1}{2} g t^2 = \frac{g l^2}{2 c^2}.$$

e.g. $l = 1\,\text{km}$ then $h = 0.055\,\text{nm}$ on Earth, $\sim 10\,\text{m}$ on a neutron star.

Laboratory must be "small" because gravity is not constant. e.g. No single inertial frame can apply to the whole Earth.

Gravitational time dilation:

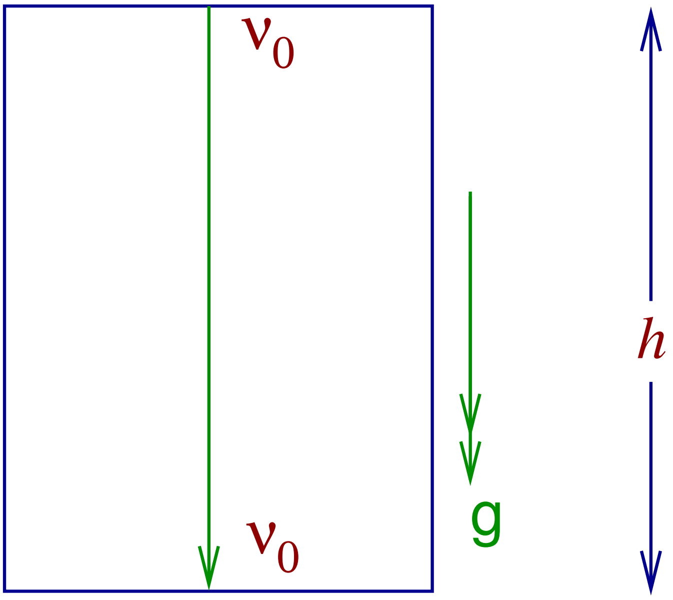

Figure: Light sent downwards in a freely-falling laboratory will be unchanged in frequency, but an Earth-based observer will see a higher freequency at the bottom since the lab is moving downwards by the time the light reaches the floor.

Assume lab is dropped at same time as light leaves ceiling. Light takes time

$$t \approx \frac{h}{c}$$

to reach floor, by which time lab is moving down at speed

$$v = \frac{gh}{c}.$$

From the EP, the frequency unchanged in lab, so according to Earth observer, the frequency at the floor is

$$\nu_1 \approx \nu_0 \left(1 + \frac{v}{c}\right) = \nu_0 \left(1 + \frac{gh}{c^2}\right) = \nu_0 \left(1 + \frac{\phi}{c^2}\right).$$

Clocks at ceiling run fast by factor $1 + \phi/c^2$ cf floor! [read up on Pound & Rebka experiment].

This "gravitational time dilation" is significant for atomic clocks on Earth.

Lecture 2

Special Relativity – I.

Objectives:

- *To recap some basic aspects of SR*
- *To introduce important notation.*

Reading: *Schutz chapter 1; Hobson chapter 1; Rindler chapter 1.*

2.1 Introduction

The equivalence principle makes Special Relativity (SR) the starting point for GR. Familiar SR equations define much of the notation used in GR.

A defining feature of SR are the Lorentz transformations (LTs), from frame S to S' which moves at v in the +ve x-direction relative to S:

$$t' = \gamma\left(t - \frac{vx}{c^2}\right), \tag{2.1}$$

$$x' = \gamma(x - vt), \tag{2.2}$$

$$y' = y, \tag{2.3}$$

$$z' = z, \tag{2.4}$$

where the Lorentz factor

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \tag{2.5}$$

Defining $x^0 = ct$, $x^1 = x$, $x^2 = y$ and $x^3 = z$, these can be re-written more symmetrically as

$$x^{0'} = \gamma\left(x^0 - \beta x^1\right), \quad (2.6)$$

$$x^{1'} = \gamma\left(x^1 - \beta x^0\right), \quad (2.7)$$

$$x^{2'} = x^2, \quad (2.8)$$

$$x^{3'} = x^3, \quad (2.9)$$

where $\beta = v/c$, so $\gamma = (1-\beta^2)^{-1/2}$.

NB. The indices here are written as superscripts; do not confuse with exponents! The dashes for the new frame are applied to the indices following Schutz.

More succinctly we have

$$x^{\alpha'} = \sum_{\beta=0}^{\beta=3} \Lambda^{\alpha'}{}_{\beta}\, x^{\beta},$$

for $\alpha' = 0$, 1, 2 or 3, where the coefficients $\Lambda^{\alpha'}{}_{\beta}$ represent the LT taking us from frame S to S'. Can write as a matrix:

$$\Lambda^{\alpha'}{}_{\beta} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (2.10)$$

with α' the row index and β the column index. Better still, using Einstein's summation convention write simply:

$$\boxed{x^{\alpha'} = \Lambda^{\alpha'}{}_{\beta}\, x^{\beta}.} \quad (2.11)$$

NB. The summation convention here is special: summation implied only when the repeated index appears once up, once down. The LT coefficients $\Lambda^{\alpha'}{}_{\beta}$ have been carefully written with a subscript to allow this. This helps keep track of indices by making some expressions, e.g. $\Lambda^{\alpha'}{}_{\beta}\, x^{\alpha'}$, invalid.

LT from S' to S is easily seen to be

$$\boxed{x^{\alpha} = \Lambda^{\alpha}{}_{\beta'}\, x^{\beta'},} \quad (2.12)$$

where

$$\Lambda^{\alpha}{}_{\beta'} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (2.13)$$

It is easily shown that

Prove this.

$$\begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Defining the Kronecker delta $\delta^\alpha_\beta = 1$ if $\alpha = \beta$, $= 0$ otherwise, this equation can be written:

$$\boxed{\Lambda^\alpha{}_{\gamma'}\Lambda^{\gamma'}{}_\beta = \delta^\alpha_\beta.} \tag{2.14}$$

Guarantees that after LTs from S to S' then back to S we get x^α again since

$$\Lambda^\alpha{}_{\gamma'}\Lambda^{\gamma'}{}_\beta\, x^\beta = \delta^\alpha_\beta x^\beta = x^\alpha.$$

Prove each step of this equation.

Note the use of dummy index γ' to avoid a clash with α or β.

2.2 Nature of LTs

In SR the coefficients of the LT are constant and thus

$$x^{\alpha'} = \Lambda^{\alpha'}{}_\beta\, x^\beta,$$

is a linear transform, mathematically very similar to spatial rotations such as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where $c = \cos\theta$, $s = \sin\theta$, $c^2 + s^2 = 1$. A defining feature of rotations is that lengths are preserved, i.e.

$$l^2 = (x')^2 + (y')^2 = x^2 + y^2.$$

Q: What general linear transform

$$\begin{aligned} x' &= \alpha x + \beta y, \\ y' &= \gamma x + \delta y, \end{aligned}$$

where α, β, γ and δ are constants, preserves lengths?

Since

$$(x')^2 + (y')^2 = \left(\alpha^2 + \gamma^2\right) x^2 + 2\left(\alpha\beta + \gamma\delta\right) xy + \left(\beta^2 + \delta^2\right) y^2,$$

then

$$\begin{aligned} \alpha^2 + \gamma^2 &= 1, \\ \alpha\beta + \gamma\delta &= 0, \\ \beta^2 + \delta^2 &= 1. \end{aligned}$$

These are satisfied by $\gamma = -\beta$ and $\delta = \alpha$, so

$$\begin{aligned} x' &= \alpha x + \beta y, \\ y' &= -\beta x + \alpha y, \end{aligned}$$

with $\alpha^2 + \beta^2 = 1$.

> Thus the requirement to preserve length defines the linear transform representing rotations.

The "interval"

$$s^2 = (ct)^2 - x^2 - y^2 - z^2,$$

plays the same role in SR.

Lecture 3

Special Relativity – II.

Objectives:

- *Four vectors*

Reading: *Schutz chapter 2, Rindler chapter 5, Hobson chapter 5*

3.1 The interval of SR

To cope with shifts of origin, restrict to the interval between two events

$$\Delta s^2 = (ct_2 - ct_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 ,$$

or

$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2,$$

or finally with infinitesimals:

$$\boxed{ds^2 = c^2\, dt^2 - dx^2 - dy^2 - dz^2.} \tag{3.1}$$

ds^2 is the same in all inertial frames. It is a Lorentz scalar. Writing

$$ds^2 = c^2\, d\tau^2,$$

defines the "proper time" τ, which is the same as the coordinate time t when $dx = dy = dz = 0$. i.e. proper time is the time measured on a clock travelling with an object.

Introducing $x^0 = ct$, etc again, we can write

$$\boxed{ds^2 = c^2\, d\tau^2 = \eta_{\alpha\beta}\, dx^\alpha\, dx^\beta,} \tag{3.2}$$

where

$$\eta_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{3.3}$$

The interval is the SR equivalent of length corresponding to the relation for lengths in Euclidean 3D

$$dl^2 = dx^2 + dy^2 + dz^2.$$

NB There is no standard sign convention for the interval and $\eta_{\alpha\beta}$. Make sure you know the convention used in textbooks.

3.2 The grain of SR

The minus signs in the definition of ds^2 means there are three types of interval:

$ds^2 > 0$ timelike intervals. Intervals between events on the wordlines of massive particles are timelike.

$ds^2 = 0$ Null intervals. Intervals between events on the wordlines of massless particles (photons) are null.

$ds^2 < 0$ Spacelike intervals which connect events out of causal contact.

These impose a distinct structure on spacetime.

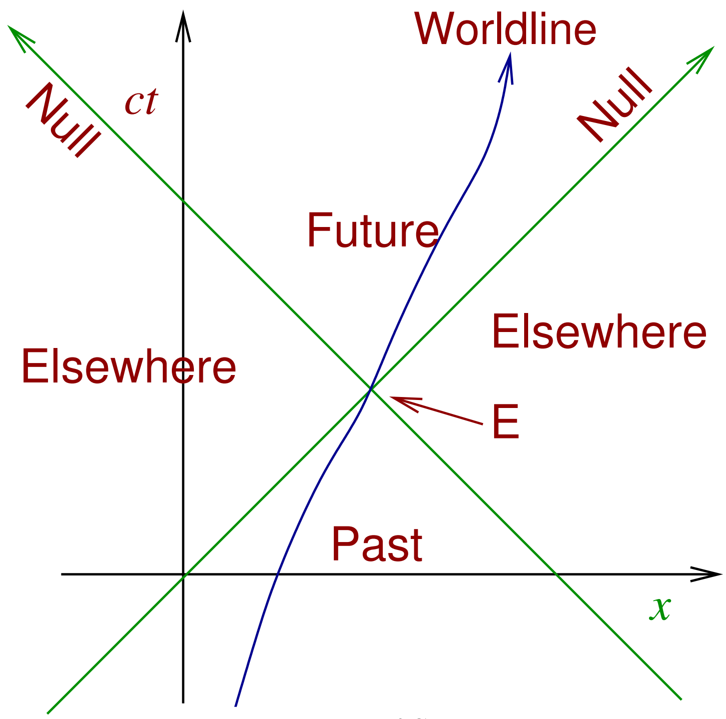

Figure: The invariant interval of SR slices up spacetime relative to an event E into past, future and "elsewhere', the latter being the events not causally connected to E.

These so-called "light-cones" are preserved in GR but are distorted according to the coordinates used.

3.3 Four-vectors

Any quantity that transforms in the same way as $\vec{X} = (x^0, x^1, x^2, x^3)$ is called a "four vector" (or often just a "vector"). Thus $\vec{V}$ is defined to be a vector if and only if

$$V^{\alpha'} = \Lambda^{\alpha'}{}_{\beta} V^{\beta}.$$

Useful because:

- Four vectors can often be identified easily
- The way they transform follows from the LTs.
- Lead to Lorentz scalars equivalent to ds^2.

3.3.1 Four-velocity

The four-velocity is one of the most important four-vectors. Consider

$$\vec{U} = \lim_{\delta\tau \to 0} \frac{\vec{X}(\tau + \delta\tau) - \vec{X}(\tau)}{\delta\tau} = \frac{d\vec{X}}{d\tau}.$$

Since $\vec{x}$ is a four-vector and τ is a scalar, $\vec{U}$ is clearly a four-vector.

From time dilation, $d\tau = dt/\gamma$, so

$$\vec{U} = \gamma \frac{d\vec{X}}{dt} = \gamma(c, \mathbf{v}),$$

where $\mathbf{v}$ is the normal three-velocity and is shorthand for the spatial components of the four-velocity.

3.3.2 Scalars from four-vectors

If $\vec{V}$ is a four-vector, then the equivalent of the interval $ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta$ is

$$\boxed{\vec{V} \cdot \vec{V} = |\vec{V}|^2 = \eta_{\alpha\beta} V^\alpha V^\beta} \tag{3.4}$$

This defines the invariant "length" or "modulus" of a four-vector. It is a scalar under LTs.

> This relation is fundamental. Note that $|\vec{V}|^2 \neq (V^0)^2 + (V^1)^2 + (V^2)^2 + (V^3)^2$. SR and GR are not Euclidean.

Example 3.1 *Calculate the scalar equivalent to the four-velocity $\vec{U}$.*

Answer 3.1 *Long way*

$$\begin{aligned}
\eta_{\alpha\beta} U^\alpha U^\beta &= \left(U^0\right)^2 - \left(U^1\right)^2 - \left(U^2\right)^2 - \left(U^3\right)^2, \\
&= \gamma^2 \left(c^2 - v_x^2 - v_y^2 - v_z^2\right), \\
&= \gamma^2 \left(c^2 - v^2\right), \\
&= \gamma^2 \frac{c^2}{\gamma^2} = c^2.
\end{aligned}$$

Short way: since it is invariant, calculate its value in a frame for which $\mathbf{v} = 0$ *and* $\gamma = 1$*, from which immediately* $\vec{U} \cdot \vec{U} = c^2$.

> $\vec{U} \cdot \vec{U} = c^2$ is an important relation. It means that $\vec{U}$ is a <u>timelike</u> four-vector.

Lecture 4

Vectors

Objectives:

- *Contravariant and covariant vectors, one-forms.*

Reading: *Schutz chapter 3; Hobson chapter 3*

4.1 Scalar or "dot" product

We have had

$$\vec{V} \cdot \vec{V} = \eta_{\alpha\beta} V^\alpha V^\beta.$$

If $\vec{A}$ and $\vec{B}$ are four-vectors then $\vec{V}$ with components

$$V^\alpha = A^\alpha + B^\alpha,$$

is also a four-vector. Therefore

$$\begin{aligned}
\vec{V} \cdot \vec{V} &= \eta_{\alpha\beta} \left(A^\alpha + B^\alpha\right) \left(A^\beta + B^\beta\right), \\
&= \eta_{\alpha\beta} A^\alpha A^\beta + \eta_{\alpha\beta} A^\alpha B^\beta + \eta_{\alpha\beta} B^\alpha A^\alpha + \eta_{\alpha\beta} B^\alpha B^\beta, \\
&= \vec{A} \cdot \vec{A} + \vec{A} \cdot \vec{B} + \vec{B} \cdot \vec{A} + \vec{B} \cdot \vec{B}.
\end{aligned}$$

Since $\eta_{\alpha\beta}$ is symmetric then $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$, so

$$\vec{V} \cdot \vec{V} = \vec{A} \cdot \vec{A} + 2\vec{A} \cdot \vec{B} + \vec{B} \cdot \vec{B}.$$

Since $\vec{V} \cdot \vec{V}$, $\vec{A} \cdot \vec{A}$ and $\vec{B} \cdot \vec{B}$ are all scalars, then

$$\vec{A} \cdot \vec{B} = \eta_{\alpha\beta} A^\alpha B^\beta \tag{4.1}$$

is also a scalar, i.e. invariant between all inertial frames. This defines the scalar product of two vectors.

$\vec{A} \cdot \vec{B} = 0 \implies \vec{A}$ and $\vec{B}$ orthogonal. Null vectors are self-orthogonal.

4.2 Basis vectors

With the following basis vectors (4D versions of $\vec{i}$, $\vec{j}$, $\vec{k}$):

$$\begin{aligned}\vec{e}_0 &= (1,0,0,0),\\ \vec{e}_1 &= (0,1,0,0),\\ \vec{e}_2 &= (0,0,1,0),\\ \vec{e}_3 &= (0,0,0,1),\end{aligned}$$

we can write for frames S and S':

$$\vec{A} = A^\alpha \vec{e}_\alpha = A^{\alpha'} \vec{e}_{\alpha'}.$$

These express the frame-independent nature of any four-vector, just as we write **a** to represent a three-vector.

Note that indices are lowered on basis vectors to fit raised indices on components.

Substituting

$$A^\alpha = \Lambda^\alpha{}_{\beta'} A^{\beta'},$$

then

$$\Lambda^\alpha{}_{\beta'} A^{\beta'} \vec{e}_\alpha = A^{\alpha'} \vec{e}_{\alpha'},$$

and re-labelling dummy indices, $\beta' \to \alpha'$, $\alpha \to \beta$,

$$\left(\vec{e}_{\alpha'} - \Lambda^\beta{}_{\alpha'} \vec{e}_\beta\right) A^{\alpha'} = 0.$$

Since $\vec{A}$ is arbitrary, the term in brackets must vanish, i.e.

$$\vec{e}_{\alpha'} = \Lambda^\beta{}_{\alpha'} \vec{e}_\beta. \tag{4.2}$$

Comparing with

$$A^{\alpha'} = \Lambda^{\alpha'}{}_\beta A^\beta,$$

we see that the components transform "oppositely" to the basis vectors, hence these are often called "contravariant vectors" and superscripted indices are called "contravariant indices".

4.3 "Covariant" vectors or "one-forms"

Consider the gradient $\nabla\phi = (\partial\phi/\partial x^0, \partial\phi/\partial x^1, \partial\phi/\partial x^2, \partial\phi/\partial x^3)$, where ϕ is a scalar function of the coordinates. Is it a vector?

The chain rule gives

$$d\phi = \frac{\partial\phi}{\partial x^\beta} dx^\beta,$$

and on differentiating wrt $x^{\alpha'}$

$$\frac{\partial \phi}{\partial x^{\alpha'}} = \frac{\partial \phi}{\partial x^{\beta}} \frac{\partial x^{\beta}}{\partial x^{\alpha'}}.$$

But $x^{\beta} = \Lambda^{\beta}{}_{\gamma'} x^{\gamma'}$ so

$$\frac{\partial x^{\beta}}{\partial x^{\alpha'}} = \Lambda^{\beta}{}_{\gamma'} \, \delta^{\gamma'}_{\alpha'} = \Lambda^{\beta}{}_{\alpha'}.$$

Therefore

$$\frac{\partial \phi}{\partial x^{\alpha'}} = \Lambda^{\beta}{}_{\alpha'} \frac{\partial \phi}{\partial x^{\beta}} \quad (4.3)$$

Thus the components of the gradient $\nabla\phi$ do not transform like the components of four-vectors, instead they transform like basis vectors.

Quantities like $\nabla\phi$ are called "covariant vectors" or "covectors" or "one-forms", the latter emphasizing their difference from vectors.

I will write one-forms with tildes such as $\tilde{p}$. Like vectors, one-forms can be defined by their transformation, i.e. if quantities p_{α} transform as

$$p_{\alpha'} = \Lambda^{\beta}{}_{\alpha'} \, p_{\beta}. \quad (4.4)$$

then they are components of a one-form $\tilde{p}$.

One-forms are written with subscripted indices, also known as "covariant" indices. Do not confuse with "Lorentz covariance".

Given a one-form $\tilde{p}$ and a vector $\vec{A}$, consider the quantity:

$$p_{\alpha} A^{\alpha}.$$

$p_{\alpha} A^{\alpha}$ is one number. Why?

Because of the "contra" and "co" transformations, this is a scalar. In a more frame-independent way we can write this as $\tilde{p}(\vec{A})$. Thus a one-form is a "machine" that produces a scalar from a vector. Equally, a vector is a machine that produces a scalar from a one-form, $\vec{A}(\tilde{p})$.

One-forms are best thought of as a series of parallel surfaces. The number of such surfaces crossed by a vector is the scalar. One-forms cannot be thought as "arrows" because they do not transform in the same way as vectors. One-forms do not crop up in orthonormal bases (e.g. Cartesian coordinates or unit vectors in polar coordinates $\hat{\mathbf{r}}$, $\hat{\theta}$*) because in that one case they transform identically to vectors. They cannot be avoided in GR.*

4.4 Basis one-forms

A set of basis vectors $\vec{e}_\beta$ define a natural set of basis one-forms $\tilde{\omega}^\alpha$:

$$\tilde{\omega}^\alpha \left(\vec{e}_\beta\right) = \delta^\alpha_\beta, \tag{4.5}$$

because then

$$\begin{aligned} \tilde{p}(\vec{A}) &= [p_\alpha \tilde{\omega}^\alpha] \left(A^\beta \vec{e}_\beta\right), \\ &= p_\alpha A^\beta \tilde{\omega}^\alpha \left(\vec{e}_\beta\right), \\ &= p_\alpha A^\beta \delta^\alpha_\beta, \\ &= p_\alpha A^\alpha, \end{aligned}$$

as required.

One can then show that basis one-forms transform like vector components, i.e.

$$\boxed{\tilde{\omega}^{\alpha'} = \Lambda^{\alpha'}{}_\beta \, \tilde{\omega}^\beta.} \tag{4.6}$$

4.5 Summary of transformations

Given a vector $\vec{A} = A^\alpha \vec{e}_\alpha$ and one-form $\tilde{p} = p_\alpha \tilde{\omega}^\alpha$ the four transformations are:

$$\begin{aligned} A^{\alpha'} &= \Lambda^{\alpha'}{}_\beta \, A^\beta, \\ \tilde{\omega}^{\alpha'} &= \Lambda^{\alpha'}{}_\beta \, \tilde{\omega}^\beta, \\ p_{\alpha'} &= \Lambda^\beta{}_{\alpha'} \, p_\beta, \\ \vec{e}_{\alpha'} &= \Lambda^\beta{}_\alpha \, \vec{e}_\beta. \end{aligned}$$

As long as you remember that vector components have superscripted indices and one-form components have subscripted indices, and balance free and dummy indices properly, it should be straightforward to remember these relations.

Lecture 5

Tensors

Objectives:

- *Introduction to tensors, the metric tensor, index raising and lowering and tensor derivatives.*

Reading: *Schutz, chapter 3; Hobson, chapter 4; Rindler, chapter 7*

5.1 Tensors

Not all physical quantities can be represented by scalars, vectors or one-forms. We will need something more flexible, and tensors fit the bill.

Tensors are "machines" that produce scalars when operating on multiple vectors and one-forms. More specifically an $\begin{pmatrix} N \\ M \end{pmatrix}$ tensor produces a scalar given N one-form and M vector arguments.

e.g. if $T(\tilde{p}, \vec{V}, \tilde{q}, \tilde{r})$ is a scalar then T is a $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ tensor.

Since vectors acting on one-forms produce scalars, vectors are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ tensors; similarly one-forms are $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ tensors and scalars are $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ tensors.

5.2 Tensor components

Components of a tensor in a given frame are found by feeding it basis vectors and one-forms. e.g.

$$T(\tilde{\omega}^\alpha, \vec{e}_\beta, \tilde{\omega}^\gamma, \tilde{\omega}^\delta) = T^\alpha{}_\beta{}^{\gamma\delta}.$$

(NB 3 up indices, 1 down matching the rank.) However, like vectors and one-forms, T exists independently of coordinates.

It is straightforward to show that for arbitrary arguments

$$T(\tilde{p}, \vec{A}, \tilde{q}, \tilde{r}) = T^\alpha{}_\beta{}^{\gamma\delta} p_\alpha A^\beta q_\gamma r_\delta.$$

All indices are dummy, so this is a single number.

For it to be a <u>scalar</u> the tensor components must transform appropriately. Using transformation properties of p_α, A^β etc, one can show that

$$T^{\alpha'}{}_{\beta'}{}^{\gamma'\delta'} = \Lambda^{\alpha'}{}_\alpha \Lambda^\beta{}_{\beta'} \Lambda^{\gamma'}{}_\gamma \Lambda^{\delta'}{}_\delta \, T^\alpha{}_\beta{}^{\gamma\delta}.$$

Extends in an obvious manner for different indices. This is often used as the <u>definition</u> of tensors, similar to our definition of vectors.

5.3 Why tensors?

Consider a $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ tensor such that $T(\vec{V}, \tilde{p})$ is a scalar. Now consider

$$T(\vec{V}, \quad),$$

i.e. one unfilled slot is available for a one-form, with which it will give a scalar $\Longrightarrow$ this is a <u>vector</u>, i.e.

$$\vec{W} = T(\vec{V}, \quad),$$

or in component form

$$W^\alpha = T_\beta{}^\alpha V^\beta.$$

This is one reason why tensors appear in physics, e.g. to relate **D** to **E** in EM, or stress to strain in solids. More importantly:

> Tensors allow us to express mathematically the frame-invariance of physical laws. If S and T are tensors and $S^{\alpha\beta} = T^{\alpha\beta}$ is true in one frame, then it is true in <u>all</u> frames.

5.4 The metric tensor

Recall the scalar product

$$\vec{A} \cdot \vec{B} = \eta_{\alpha\beta} \, A^\alpha \, B^\beta.$$

$\vec{A} \cdot \vec{B}$ is a scalar while $\vec{A}$ and $\vec{B}$ are vectors. η is therefore a $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor producing a scalar given two vector arguments:

$$\vec{A} \cdot \vec{B} = \eta \left(\vec{A}, \vec{B} \right).$$

$\eta_{\alpha\beta}$ are thus components of a tensor, the "metric tensor".

5.4.1 Index raising and lowering

The metric tensor arises directly from the physics of spacetime. This gives it a special place in associating vectors and one-forms. Consider as before an unfilled slot, this time with η:

$$\eta(\vec{A}, \ \ ).$$

Fed a vector, this returns a scalar, so it is a one-form. We define this as the one-form equivalent to the vector $\vec{A}$:

$$\tilde{A} = \eta(\vec{A}, \ \ ),$$

or in component form

$$A_\alpha = \eta_{\alpha\beta} A^\beta.$$

Thus $\eta_{\alpha\beta}$ can be used to lower indices, as in

$$T_{\alpha\beta} = \eta_{\alpha\gamma} T^\gamma{}_\beta,$$

or

$$T_{\alpha\beta} = \eta_{\alpha\gamma}\eta_{\beta\delta} T^{\gamma\delta}.$$

If we define $\eta^{\alpha\beta}$ by

$$\eta^{\alpha\gamma}\eta_{\gamma\beta} = \delta^\alpha_\beta,$$

then applying it to an arbitrary one-form

In SR $\eta^{\alpha\beta} = \eta_{\alpha\beta}$.

$$\begin{aligned} \eta^{\alpha\gamma} A_\gamma &= \eta^{\alpha\gamma} \left(\eta_{\gamma\delta} A^\delta \right), \\ &= \left(\eta^{\alpha\gamma} \eta_{\gamma\delta} \right) A^\delta, \\ &= \delta^\alpha_\delta A^\delta, \\ &= A^\alpha, \end{aligned}$$

so it raises indices.

The metric tensor in its covariant and contravariant forms, $\eta_{\alpha\beta}$ and $\eta^{\alpha\beta}$, can be used to switch between one-forms and vectors and to lower or raise any given index of a tensor.

e.g. $\eta^{\alpha\beta}\partial\phi/\partial x^\beta$ is a gradient vector.

5.5 Derivatives of tensors

Derivatives of scalars, such as $\partial\phi/\partial x^\alpha = \partial_\alpha \phi$ give one-forms but what about derivatives of vectors, $\partial V^\beta/\partial x^\alpha$?

Work out how they transform:

$$V^{\beta'} = \Lambda^{\beta'}{}_{\gamma} V^\gamma$$

thus

$$\begin{aligned} \frac{\partial V^{\beta'}}{\partial x^{\alpha'}} &= \frac{\partial}{\partial x^{\alpha'}} \left[\Lambda^{\beta'}{}_{\gamma} V^\gamma\right], \\ &= \Lambda^{\beta'}{}_{\gamma} \frac{\partial V^\gamma}{\partial x^{\alpha'}}, \end{aligned}$$

because the $\Lambda^{\beta'}{}_{\gamma}$ are constant in SR (but not in GR!).

Using the chain-rule

$$\frac{\partial}{\partial x^{\alpha'}} = \frac{\partial x^\delta}{\partial x^{\alpha'}} \frac{\partial}{\partial x^\delta},$$

and as in the last lecture

$$\frac{\partial x^\delta}{\partial x^{\alpha'}} = \Lambda^{\delta}{}_{\alpha'}.$$

Therefore

$$\frac{\partial V^{\beta'}}{\partial x^{\alpha'}} = \Lambda^{\beta'}{}_{\gamma} \Lambda^{\delta}{}_{\alpha'} \frac{\partial V^\gamma}{\partial x^\delta}.$$

This is the transformation rule of a $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ tensor. Key point:

The derivatives of tensors are also tensors – we don't need to introduce a new type of quantity – phew!

Lecture 6

Stress–energy tensor

Objectives:

- *To introduce the stress–energy tensor*
- *Conservation laws in relativity*

Reading: *Schutz chapter 4; Hobson, chapter 8; Rindler, chapter 7.*

6.1 Number–flux vector

Consider a cloud of particles ("dust") at rest in frame S_0, the "instantaneous rest frame" or IRF with number density n_0.

Lorentz contraction means that a cube dx_0, dy_0, dz_0 in S_0 transforms to $dx = dx_0/\gamma$, $dy = dy_0$, $dz = dz_0$ in a frame S in which the particles move, while particle numbers are conserved, so in S the particle density n is given by

$$n = \gamma n_0.$$

n is not a scalar or a four-vector and so cannot be part of form-invariant relations. Consider instead

$$\vec{N} = n_0 \vec{U}.$$

This is a four-vector because

- The four velocity $\vec{U} = \gamma(c, \mathbf{v})$ is a four-vector
- n_0 is a scalar (defined in the IRF so all observers agree on it).

The time component $N^0 = \gamma n_0 c = nc$ gives the number density. The spatial components $N^i = \gamma n_0 v^i = nv^i$, $i = 1, 2, 3$ are the fluxes (particles/unit area/unit time) across surfaces of constant x, y and z.

Even N^0 is a "flux across a surface", a surface of constant time:

Sketch this:

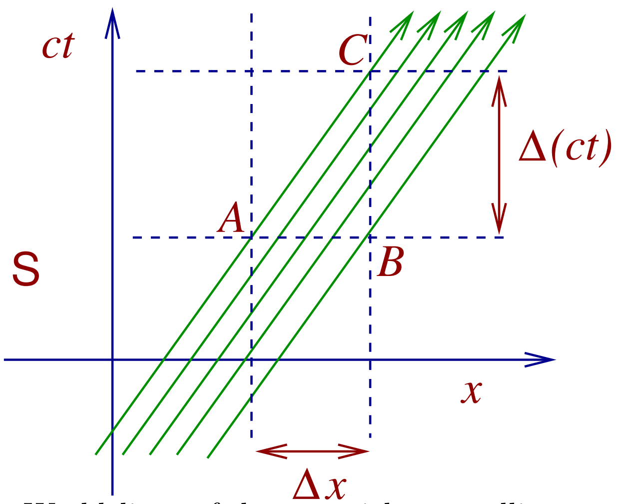

Figure: World lines of dust particles travelling at speed v in the x-direction crossing surfaces of constant t (A–B) and constant x (B–C).

Worldlines crossing CB represent the flux across constant x, $N^1 = nv$

Same worldlines crossing AB represent flux across constant t. Scaling by ratio of sides of triangle we get a flux:

$$N^1 \frac{CB}{AB} = N^1 \frac{\Delta(ct)}{\Delta x} = N^1 \frac{c}{v} = N^0,$$

so N^0 is the particle flux across a surface of constant time.

6.2 Conservation of particle numbers

Consider the scalar $\tilde{\nabla}(\vec{N})$ (one-form $\tilde{\nabla}$ acting on $\vec{N}$). Written out in full:

$$\begin{aligned}\tilde{\nabla}(\vec{N}) &= \frac{\partial N^\alpha}{\partial x^\alpha}, \\ &= \frac{\partial N^0}{\partial x^0} + \frac{\partial N^1}{\partial x^1} + \frac{\partial N^2}{\partial x^2} + \frac{\partial N^3}{\partial x^3}, \\ &= \frac{\partial nc}{\partial ct} + \frac{\partial nv_x}{\partial x} + \frac{\partial nv_y}{\partial y} + \frac{\partial nv_z}{\partial z}.\end{aligned}$$

This can be written as

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}).$$

Compare with the continuity equation of fluid mechanics:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{v}) = 0,$$

based on (Newtonian) conservation of mass . $\implies$ if particles are conserved:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0.$$

Thus conservation of particle numbers can be expressed as:

$$\boxed{\tilde{\nabla}(\vec{N}) = \frac{\partial N^\alpha}{\partial x^\alpha} = \partial_\alpha N^\alpha = N^\alpha{}_{,\alpha} = 0,} \tag{6.1}$$

introducing the short-hand $\partial_\alpha = \partial/\partial x^\alpha$, and the even shorter-hand comma notation for derivatives.

6.3 Stress–energy tensor

If the mass density in the IRF is ρ_0, then due to Lorentz contraction and relativistic mass increase, in any other frame it becomes:

$$\rho = \gamma^2 \rho_0,$$

Now consider

$$T^{\alpha\beta} = \rho_0 U^\alpha U^\beta,$$

then since $U^0 = \gamma c$,

$$T^{00} = \gamma^2 \rho_0 c^2 = \rho c^2.$$

From $E = mc^2$, T^{00} must therefore be the energy density.

T is a tensor because

- The four velocity $\vec{U}$ is a four-vector
- ρ_0 is a scalar (defined in the IRF)

T is called the stress–energy tensor.

6.3.1 Physical meaning

$T^{\alpha\beta}$ is the flux of the α-th component of four-momentum across a surface of constant x^β, so:

- T^{00} = flux of 0-th component of four-momentum (energy) across the time surface (cf N^0) = energy density
- $T^{0i} = T^{i0}$ = energy flux across surface of constant x^i (heat conduction in IRF)
- T^{ij} = flux of i-momentum across j surface = "stress".

6.4 Perfect fluids

Definition: a perfect fluid has (i) no heat conduction and (ii) no viscosity.

In the IRF (i) implies $T^{0i} = T^{i0} = 0$, while (ii) implies $T^{ij} = 0$ if $i \neq j$.

For T^{ij} to be diagonal for any orientation of axes $\Longrightarrow$ $T^{ij} = p_0 \delta^{ij}$ where p_0 is the pressure in the IRF. Therefore in the IRF:

Convince yourself of this.

$$T^{\alpha\beta} = \begin{pmatrix} \rho_0 c^2 & 0 & 0 & 0 \\ 0 & p_0 & 0 & 0 \\ 0 & 0 & p_0 & 0 \\ 0 & 0 & 0 & p_0 \end{pmatrix}.$$

But this can be written:

$$\boxed{T^{\alpha\beta} = \left(\rho_0 + \frac{p_0}{c^2}\right) U^\alpha U^\beta - p_0 \eta^{\alpha\beta},}$$

and since all terms are tensors, this is true in any frame remembering that ρ_0 and p_0 are defined in the IRF.

The sign of the p_0 term can vary according to convention adopted for η

Just as conservation of particles implies $N^\alpha{}_{,\alpha} = 0$, so energy–momentum conservation gives

$$\boxed{T^{\alpha\beta}{}_{,\beta} = \frac{\partial T^{\alpha\beta}}{\partial x^\beta} = 0.}$$

This equation plays a key role in GR where the stress–energy tensor replaces the simple density, ρ, of Newtonian gravity.

Lecture 7

Generalised Coordinates

Objectives:

- *Generalised coordinates*
- *Transformations between coordinates*

Reading: *Schutz, 5 and 6; Hobson, 2; Rindler, 8.*

Consider the following situation:

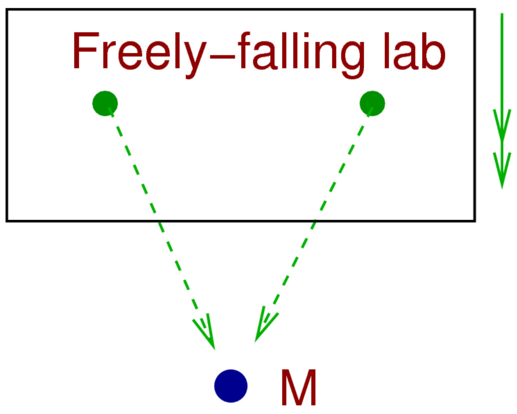

Figure: A freely falling laboratory with two small masses floating within it.

Lab falls freely with two small masses within it. The masses accelerate towards centre of mass M. Therefore they will end up moving towards each other.

The masses can be made as small as one likes, so their movement is not because of their mutual gravitational attraction.

Equivalence principle says SR in a small freely-falling lab, but clearly not true over large region.

Einstein's remarkable insight was that this was similar to the following:

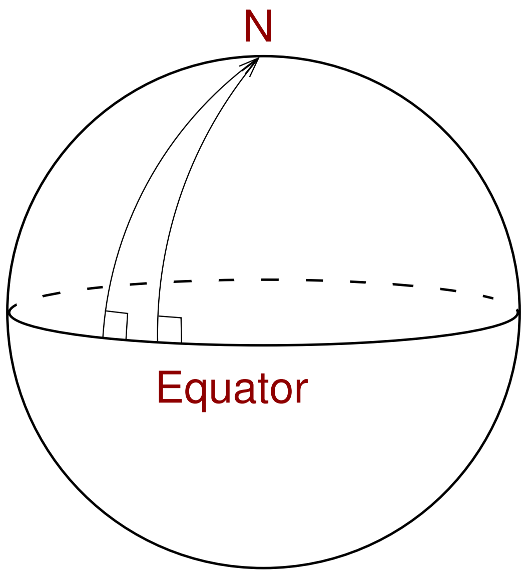

Figure: Two people set off due North from the equator on Earth.

Two people at Earth's equator travel due North, i.e. parallel to each other. Although they stick to "straight" paths, they find that they move towards each other, and ultimately meet at the North pole.

> Einstein replaced Newtonian gravity by the <u>curvature</u> of spacetime. Although particles travel in straight lines in spacetime, the warping of spacetime by large masses can cause initially parallel paths to converge. *There is no gravitational force in GR!*

7.1 Coordinates

We have to be able to cope with general coordinates covering potentially curved spaces $\implies$ <u>differential geometry</u> developed by Gauss, Riemann and many others.

Start by defining a set of coordinates covering an N-dimensional space ("manifold") by x^1, x^2, x^3, $\ldots x^N$. [Temporary suspension of 0 index to avoid $N-1$ everywhere.]

7.2 Curves

A curve can be defined by the N parametric equations

$$x^\alpha = x^\alpha(\lambda),$$

for each α, where λ is a parameter marking position along the curve. e.g. $x = \lambda$, $y = \lambda^2$ is a parabola in 2D. λ independent of coordinates $\Longrightarrow$ scalar.

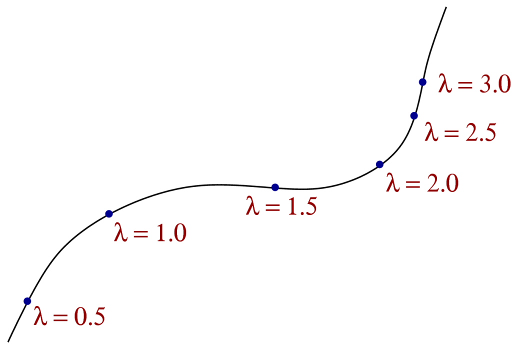

Figure: A curve parameterised by parameter λ.

7.3 Coordinate transforms

Coordinates can always be re-labelled:

$$x^{\alpha'} = x^{\alpha'}(x^1, x^2, \ldots, x^\beta, \ldots x^N),$$

or $x^{\alpha'} = x^{\alpha'}(x^\beta)$ for short. This is a coordinate transformation.

Example 7.1 *In Euclidean 2D*

$$\begin{aligned} r &= \left(x^2 + y^2\right)^{1/2}, \\ \theta &= \cos^{-1}(x/r), \end{aligned}$$

transforms from Cartesian to polar coordinates.

Recall the SR equation:

$$x^{\alpha'} = \Lambda^{\alpha'}{}_{\beta}\, x^\beta.$$

Compare with:

$$dx^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\beta}\, dx^\beta,$$

then the $N \times N$ partial derivatives $\partial x^{\alpha'}/\partial x^\beta$ define the transformation matrix:

$$\mathbf{L} = \begin{pmatrix} \partial x^{1'}/\partial x^1 & \partial x^{1'}/\partial x^2 & \ldots & \partial x^{1'}/\partial x^N \\ \partial x^{2'}/\partial x^1 & \partial x^{2'}/\partial x^2 & \ldots & \partial x^{2'}/\partial x^N \\ \vdots & \vdots & \vdots & \vdots \\ \partial x^{N'}/\partial x^1 & \partial x^{N'}/\partial x^2 & \ldots & \partial x^{N'}/\partial x^N \end{pmatrix},$$

a generalisation of the LT matrix $\mathbf{\Lambda}$. The $L^{\alpha'}{}_{\beta}$ are not constant unlike $\Lambda^{\alpha'}{}_{\beta}$ in SR; the transformation also only applies to infinitesimal displacements.

> Good news: With $\partial x^{\alpha'}/\partial x^{\beta}$ instead of $\Lambda^{\alpha'}{}_{\beta}$, the transformation formulae for vectors, one-forms and tensors are otherwise unchanged.

7.4 The general metric tensor

In a freely-falling frame (SR), let coordinates be w^{α}, so the interval is

$$ds^2 = \eta_{\gamma\delta}\, dw^{\gamma}\, dw^{\delta}.$$

Replacing w with x using

$$dw^{\gamma} = \frac{\partial w^{\gamma}}{\partial x^{\alpha}}\, dx^{\alpha} \text{ and } dw^{\delta} = \frac{\partial w^{\delta}}{\partial x^{\beta}}\, dx^{\beta},$$

avoiding clashing indices, gives

$$ds^2 = \eta_{\gamma\delta} \frac{\partial w^{\gamma}}{\partial x^{\alpha}} \frac{\partial w^{\delta}}{\partial x^{\beta}}\, dx^{\alpha}\, dx^{\beta}.$$

Setting

$$g_{\alpha\beta} = \eta_{\gamma\delta} \frac{\partial w^{\gamma}}{\partial x^{\alpha}} \frac{\partial w^{\delta}}{\partial x^{\beta}},$$

we therefore have the very important relation

$$\boxed{ds^2 = g_{\alpha\beta}\, dx^{\alpha}\, dx^{\beta}.} \qquad (7.1)$$

$g_{\alpha\beta}$ is the generalised version of the SR metric tensor $\eta_{\alpha\beta}$ and replaces it.

e.g. In general coordinates, the four-velocity $\vec{U}$ satisfies

$$\boxed{g_{\alpha\beta} U^{\alpha} U^{\beta} = c^2.} \qquad (7.2)$$

> The first part of the transition from SR to GR is to replace every occurrence of $\eta_{\alpha\beta}$ by $g_{\alpha\beta}$.

e.g. index raising lowering $A_{\alpha} = \eta_{\alpha\beta} A^{\beta}$ becomes $A_{\alpha} = g_{\alpha\beta} A^{\beta}$. $g_{\alpha\beta}$ is symmetric but not necessarily diagonal; $\eta_{\alpha\beta}$ is a special case. Similarly $\eta_{\alpha\gamma}\eta^{\gamma\beta} = \delta^{\beta}_{\alpha}$ becomes $g_{\alpha\gamma} g^{\gamma\beta} = \delta^{\beta}_{\alpha}$, so the "up" coefficients come from the matrix-inverse of the "down" ones.

Lecture 8

Metrics

Objectives:

- *More on the metric and how it transforms.*

Reading: *Hobson, 2.*

8.1 Riemannian Geometry

The interval

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta,$$

is a quadratic function of the coordinate differentials.

This is the definition of Riemannian geometry, or more correctly, pseudo-Riemannian geometry to allow for $ds^2 < 0$.

Example 8.1 *What are the coefficients of the metric tensor in 3D Euclidean space for Cartesian, cylindrical polar and spherical polar coordinates?*

Answer 8.1 *The "interval" in Euclidean geometry can be written in Cartesian coordinates as*

$$ds^2 = dx^2 + dy^2 + dz^2.$$

The metric tensor's coefficients are therefore given by

$$g_{xx} = g_{yy} = g_{zz} = 1,$$

with all others $= 0$.

Introducing an obvious notation with x standing for the x coordinate index, etc.

In cylindrical polars:

$$ds^2 = dr^2 + r^2 d\phi^2 + dz^2,$$

so $g_{rr} = 1$, $g_{\phi\phi} = r^2$, $g_{zz} = 1$ *and all others* $= 0$.

Finally spherical polars:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2,$$

gives $g_{rr} = 1$, $g_{\theta\theta} = r^2$ *and* $g_{\phi\phi} = r^2 \sin^2\theta$.

Example 8.2 *Calculate the metric tensor in 3D Euclidean space for the coordinates* $u = x + 2y$, $v = x - y$, $w = z$.

Answer 8.2 *The inverse transform is easily shown to be* $x = (u + 2v)/3$, $y = (u - v)/3$, $z = w$, *so*

$$\begin{aligned} dx &= \frac{1}{3}du + \frac{2}{3}dv, \\ dy &= \frac{1}{3}du - \frac{1}{3}dv, \\ dz &= dw, \end{aligned}$$

so

$$\begin{aligned} ds^2 &= \left(\frac{1}{3}du + \frac{2}{3}dv\right)^2 + \left(\frac{1}{3}du - \frac{1}{3}dv\right)^2 + dw^2, \\ &= \frac{2}{9}du^2 + \frac{5}{9}dv^2 + \frac{2}{9}dudv + dw^2. \end{aligned}$$

We can immediately write $g_{uu} = 2/9$, $g_{vv} = 5/9$, $g_{ww} = 1$, *and* $g_{uv} = g_{vu} = 1/9$ *since the metric is symmetric. This metric* still *describes 3D Euclidean flat geometry, although not obviously.*

8.2 Metric transforms

The method of the example is often the easiest way to transform metrics, however using tensor transformations, we can write more compactly:

$$g_{\alpha'\beta'} = \frac{\partial x^\gamma}{\partial x^{\alpha'}} \frac{\partial x^\delta}{\partial x^{\beta'}} g_{\gamma\delta}.$$

This shows how the components of the metric tensor transform under coordinate transformations but the underlying geometry does not change.

Example 8.3 *Use the transformation of g to derive the metric components in cylindrical polars, starting from Cartesian coordinates.*

Answer 8.3 *We must compute terms like $\partial x/\partial r$, so we need x, y and z in terms of r, ϕ, z:*

$$\begin{aligned} x &= r\cos\phi, \\ y &= r\sin\phi, \\ z &= z. \end{aligned}$$

Find $\partial x/\partial r = \cos\phi$, $\partial y/\partial r = \sin\phi$, $\partial z/\partial r = 0$. Consider the g_{rr} component:

$$g_{rr} = \frac{\partial x^i}{\partial r}\frac{\partial x^j}{\partial r} g_{ij},$$

where i and j represent x, y or z. Since $g_{ij} = 1$ for $i = j$ and 0 otherwise, and since $\partial z/\partial r = 0$, we are left with:

$$\begin{aligned} g_{rr} &= \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 \\ &= \cos^2\phi + \sin^2\phi = 1. \end{aligned}$$

Similarly

$$g_{\theta\theta} = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 = (-r\sin\phi)^2 + (r\cos\phi)^2 = r^2,$$

and $g_{zz} = 1$, as expected.

This may seem a very difficult way to deduce a familiar result, but the point is that it transforms a problem for which one otherwise needs to apply intuition and 3D visualisation into a mechanical procedure that is not difficult – at least in principle – and can even be programmed into a computer.

8.3 First curved-space metric

We can now start to look at curved spaces. A very helpful one is the surface of a sphere.

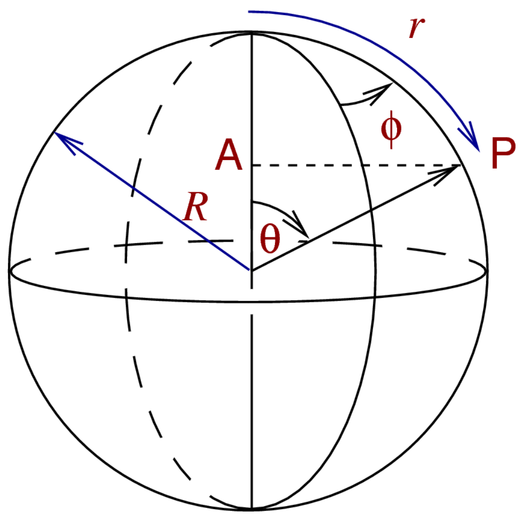

Figure: Surface of a sphere parameterised by distance r from a point and azimuthal angle ϕ

The sketch shows the surface "embedded" in 3D. This is a priviledged view that is not always possible. You need to try to imagine that you are actually stuck in the surface with no "height" dimension.

Two coordinates are needed to label the surface. e.g. the distance from a point along the surface, r, and the azimuthal angle ϕ, similar to Euclidean polar coords.

The distance AP is given by $R \sin\theta$, so a change $d\phi$ corresponds to distance $R \sin\theta \, d\phi$. Thus the metric is

$$ds^2 = dr^2 + R^2 \sin^2\theta \, d\phi^2.$$

or since $r = R\theta$,

$$ds^2 = dr^2 + R^2 \sin^2\left(\frac{r}{R}\right) d\phi^2.$$

This is the metric of a 2D space of constant curvature.

Circumference of circle in this geometry: set $dr = 0$, integrate over ϕ

$$C = 2\pi R \sin\frac{r}{R} < 2\pi r.$$

e.g. On Earth ($R = 6370\,\text{km}$), circle with $r = 10\,\text{km}$ shorter by 2.6 cm than if Earth was flat.

Exactly the same is possible in 3D. i.e we could find that a circle radius r has a circumference $< 2\pi r$ owing to gravitationally induced curvature.

8.4 2D spaces of constant curvature

Can construct metric of the surface of a sphere as follows. First write the equation of a sphere in Euclidean 3D

$$x^2 + y^2 + z^2 = R^2.$$

If we switch to polars (r, θ) in the x–y plane, this becomes

$$r^2 + z^2 = R^2.$$

In the same terms the Euclidean metric is

$$dl^2 = dr^2 + r^2 d\theta^2 + dz^2.$$

But we can use the restriction to a sphere to eliminate dz which implies

$$2r\,dr + 2z\,dz = 0,$$

and so

$$dl^2 = dr^2 + r^2 d\theta^2 + \frac{r^2 dr^2}{z^2},$$

which reduces to

$$dl^2 = \frac{dr^2}{1 - r^2/R^2} + r^2 d\theta^2.$$

Defining curvature $k = 1/R^2$, we have

$$dl^2 = \frac{dr^2}{1 - kr^2} + r^2 d\theta^2,$$

the metric of a 2D space of constant curvature. $k > 0$ can be "embedded" in 3D as the surface of a sphere; $k < 0$ cannot, but it is still a perfectly valid geometry. [A saddle shape has negative curvature over a limited region.]

A very similar procedure can be used to construct the spatial part of the metric describing the Universe.

Lecture 9

The connection

Objectives:

- *The connection*

Reading: *Schutz 5; Hobson 3; Rindler 10.*

Apart from the change from $\eta_{\alpha\beta}$ to its more general counterpart, $g_{\alpha\beta}$, we have not had to change much in moving from SR to more general coordinates, but this comes to an end when we look again at derivatives.

9.1 Covariant derivatives of vectors

We showed that $\partial V^\alpha/\partial x^\beta$ are components of a tensor in SR; this is not true in GR. Consider the derivative of $\vec{V} = V^\alpha \vec{e}_\alpha$:

$$\frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta}\vec{e}_\alpha + V^\alpha \frac{\partial \vec{e}_\alpha}{\partial x^\beta}.$$

$\partial \vec{e}_\alpha/\partial x^\beta$, the change in a vector is still a vector, and hence can be expanded over the basis:

$$\boxed{\frac{\partial \vec{e}_\alpha}{\partial x^\beta} = \Gamma^\gamma{}_{\alpha\beta}\, \vec{e}_\gamma} \tag{9.1}$$

where the $\Gamma^\gamma{}_{\alpha\beta}$ are a set of coefficients dependent upon position. They are called variously the "connection coefficients" or "Christoffel symbols". This equation <u>defines</u> the coefficients $\Gamma^\gamma{}_{\alpha\beta}$.

Sometimes "Christoffel symbols of the second kind"

Swapping indices α and γ, we can write

$$\boxed{\frac{\partial \vec{V}}{\partial x^\beta} = \left(\frac{\partial V^\alpha}{\partial x^\beta} + \Gamma^\alpha{}_{\gamma\beta} V^\gamma\right)\vec{e}_\alpha.} \tag{9.2}$$

The derivative of a vector must be a tensor, so

$$\frac{\partial V^\alpha}{\partial x^\beta} + \Gamma^\alpha{}_{\gamma\beta} V^\gamma,$$

are the components of a tensor, called the covariant derivative, written in frame-independent notation as $\nabla \vec{V}$ with components

$$\boxed{\nabla_\beta V^\alpha = \partial_\beta V^\alpha + \Gamma^\alpha{}_{\gamma\beta} V^\gamma.} \tag{9.3}$$

or equivalently

$$\boxed{V^\alpha{}_{;\beta} = V^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\gamma\beta} V^\gamma,} \tag{9.4}$$

introducing the semi-colon notation to represent the covariant derivative.

The final notation has the advantage that the β index is last in every term. Otherwise, try to remember that whichever component you take the derivative with respect to goes last on the connection coefficients.

The two terms $\partial_\beta V^\alpha$ and $\Gamma^\alpha{}_{\gamma\beta} V^\gamma$ are do not transform as tensors, only their sum does; in SR $\partial_\beta V^\alpha$ are tensor components while $\Gamma^\alpha{}_{\gamma\beta} = 0$.

$\partial_\beta V^\alpha$ comes from the change of components with position, $\Gamma^\alpha{}_{\gamma\beta} V^\gamma$ comes from the change of basis vectors with position.

Example 9.1 *Calculate the connection coefficients in Euclidean polar coordinates r, θ.*

Answer 9.1 *Start from Cartesian basis vectors $\vec{e}_x$ and $\vec{e}_y$. Using the transformation rule for basis vectors:*

$$\vec{e}_{\alpha'} = \frac{\partial x^\beta}{\partial x^{\alpha'}} \vec{e}_\beta,$$

we have

$$\vec{e}_r = \frac{\partial x}{\partial r} \vec{e}_x + \frac{\partial y}{\partial r} \vec{e}_y,$$

and since $x = r\cos\theta$, $y = r\sin\theta$,

$$\vec{e}_r = \cos\theta\, \vec{e}_x + \sin\theta\, \vec{e}_y.$$

Similarly

$$\vec{e}_\theta = -r\sin\theta\, \vec{e}_x + r\cos\theta\, \vec{e}_y.$$

Prove this.

Taking derivatives, remembering that the Cartesian vectors are constant, we have

$$\begin{aligned}\frac{\partial \vec{e}_r}{\partial r} &= 0,\\ \frac{\partial \vec{e}_r}{\partial \theta} &= -\sin\theta\vec{e}_x + \cos\theta\vec{e}_y,\\ \frac{\partial \vec{e}_\theta}{\partial r} &= -\sin\theta\vec{e}_x + \cos\theta\vec{e}_y,\\ \frac{\partial \vec{e}_\theta}{\partial \theta} &= -r\cos\theta\vec{e}_x - r\sin\theta\vec{e}_y,\end{aligned}$$

which we can re-write as

$$\begin{aligned}\frac{\partial \vec{e}_r}{\partial r} &= 0,\\ \frac{\partial \vec{e}_r}{\partial \theta} &= \frac{1}{r}\vec{e}_\theta,\\ \frac{\partial \vec{e}_\theta}{\partial r} &= \frac{1}{r}\vec{e}_\theta,\\ \frac{\partial \vec{e}_\theta}{\partial \theta} &= -r\vec{e}_r.\end{aligned}$$

Hence the Christoffel symbols are ${\Gamma^\theta}_{r\theta} = {\Gamma^\theta}_{\theta r} = 1/r$, ${\Gamma^r}_{\theta\theta} = -r$, *and* ${\Gamma^r}_{rr} = {\Gamma^\theta}_{rr} = {\Gamma^r}_{r\theta} = {\Gamma^r}_{\theta r} = {\Gamma^\theta}_{\theta\theta} = 0$.

Note that the final set of relations does not involve Cartesian vectors. The Christoffel symbols allow one to work in complex coordinate systems without reference to Cartesian coordinates, and to derive such well-known formulae such as the Laplacian in spherical coordinates – see Schutz or Hobson for this.

The way we calculated the connection above is tedious and indirect, but there is a better way.

9.2 The Levi-Civita Connection

One can show that See handout 3

$$\boxed{{\Gamma^\alpha}_{\gamma\beta} = \frac{1}{2} g^{\alpha\delta}\left(g_{\delta\beta,\gamma} + g_{\gamma\delta,\beta} - g_{\gamma\beta,\delta}\right),}$$

which is known as the Levi-Civita connection and shows that the connection can be calculated from the metric alone without recourse to Cartesian coordinates.

Example 9.2 *Calculate the connection coefficients in polar coordinates* (r, θ).

Answer 9.2 *The metric is* $ds^2 = dr^2 + r^2\, d\theta^2$, *so* $g_{rr} = g^{rr} = 1$, $g_{\theta\theta} = r^2$, $g^{\theta\theta} = 1/r^2$, *while all* $g_{r\theta} = 0$.

Thus

$$\begin{aligned}
\Gamma^{\theta}{}_{r\theta} &= \frac{1}{2} g^{\theta\theta} \left(g_{\theta r,\theta} + g_{\theta\theta,r} - g_{r\theta,\theta} \right), \\
&= \frac{1}{2} g^{\theta\theta} g_{\theta\theta,r}, \\
&= \frac{1}{2} \frac{1}{r^2} 2r, \\
&= \frac{1}{r}.
\end{aligned}$$

This agrees with the value found earlier, and although algebraically tricky, is more straightforward.

9.3 Covariant derivatives of one-forms

What is the equivalent for one-forms of

$$V^{\alpha}{}_{;\beta} = V^{\alpha}{}_{,\beta} + \Gamma^{\alpha}{}_{\gamma\beta} V^{\gamma}\, ?$$

Consider the scalar $\phi = p_\alpha V^\alpha$, then $\phi_{,\beta}$ is a tensor and

$$\phi_{,\beta} = p_\alpha V^{\alpha}{}_{,\beta} + p_{\alpha,\beta} V^\alpha.$$

Writing

$$\phi_{,\beta} = p_\alpha \left(V^{\alpha}{}_{,\beta} + \Gamma^{\alpha}{}_{\gamma\beta} V^\gamma \right) + \left(p_{\alpha,\beta} - \Gamma^{\gamma}{}_{\alpha\beta} p_\gamma \right) V^\alpha,$$

or

$$\phi_{,\beta} = p_\alpha V^{\alpha}{}_{;\beta} + \left(p_{\alpha,\beta} - \Gamma^{\gamma}{}_{\alpha\beta} p_\gamma \right) V^\alpha.$$

All terms outside brackets are tensors and therefore

$$\boxed{p_{\alpha;\beta} = p_{\alpha,\beta} - \Gamma^{\gamma}{}_{\alpha\beta} p_\gamma,}$$

is a tensor, the covariant derivative of the one-form.

These results generalise to general tensors, e.g.

$$T^{\alpha\beta}{}_{\gamma\delta;\sigma} = T^{\alpha\beta}{}_{\gamma\delta,\sigma} + \Gamma^{\alpha}{}_{\rho\sigma} T^{\rho\beta}{}_{\gamma\delta} + \Gamma^{\beta}{}_{\rho\sigma} T^{\alpha\rho}{}_{\gamma\delta} - \Gamma^{\rho}{}_{\gamma\sigma} T^{\alpha\beta}{}_{\rho\delta} - \Gamma^{\rho}{}_{\delta\sigma} T^{\alpha\beta}{}_{\gamma\rho}$$

i.e one +ve term for each contravariant index, one −ve term for each covariant one, derivative index always last on connection.

This chapter/lecture has introduced the important concept of the "covariant derivative" which allows us to write frame-invariant tensor derivatives in GR.

Lecture 10

Parallel transport

Objectives:

- *Parallel transport*
- *Geodesics*
- *Equations of motion*

Reading: *Schutz 6; Hobson 3; Rindler 10.*

In this lecture we are finally going to see how the metric determines the motion of particles. First we discuss the concept of "parallel transport".

10.1 Parallel transport

In SR, the equation for force-free motion of a particle is

$$\vec{A} = \frac{d\vec{U}}{d\tau} = 0,$$

i.e a straight line through spacetime as well as 3D space with the vector $\vec{U}$ remaining constant along the line parameterised by τ.

This is extended to the curved spacetime of GR by the notion of parallel "transport" in which a vector is moved along a curve staying parallel to itself and of constant magnitude.

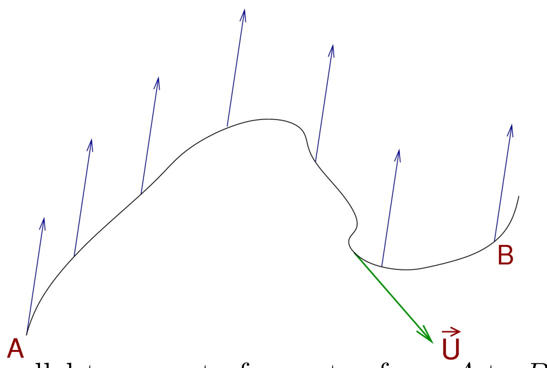

Figure: Parallel transport of a vector from A to B, keeping it parallel to itself and of constant length at all points.

Consider the change of a vector $\vec{V}$ along a line parameterised by λ

$$\frac{d\vec{V}}{d\lambda} = \frac{dV^\alpha}{d\lambda}\vec{e}_\alpha + V^\alpha \frac{d\vec{e}_\alpha}{d\lambda}.$$

We can write

$$\frac{d\vec{e}_\alpha}{d\lambda} = \frac{\partial \vec{e}_\alpha}{\partial x^\beta}\frac{dx^\beta}{d\lambda}.$$

Using this and the definition of the connection

$$\frac{\partial \vec{e}_\alpha}{\partial x^\beta} = \Gamma^\gamma{}_{\alpha\beta}\vec{e}_\gamma,$$

gives

$$\frac{d\vec{V}}{d\lambda} = \frac{dV^\alpha}{d\lambda}\vec{e}_\alpha + V^\alpha \Gamma^\gamma{}_\alpha \beta \frac{dx^\beta}{d\lambda}\vec{e}_\gamma.$$

Swapping dummy indices α and γ in the second term finally leads to

$$\boxed{\frac{d\vec{V}}{d\lambda} = \left(\frac{dV^\alpha}{d\lambda} + \Gamma^\alpha{}_{\gamma\beta}\frac{dx^\beta}{d\lambda}V^\gamma\right)\vec{e}_\alpha.}$$

This is a vector with components

$$\boxed{\frac{DV^\alpha}{D\lambda} = \frac{dV^\alpha}{d\lambda} + \Gamma^\alpha{}_{\gamma\beta}\frac{dx^\beta}{d\lambda}V^\gamma,}$$

and is known variously as the "intrinsic", "absolute" or "total" derivative. One also sometimes sees the vector written as

$$\frac{d\vec{V}}{d\lambda} = \nabla_{\vec{U}}\vec{V},$$

where $U^\alpha = dx^\alpha/d\lambda$ is the "tangent vector" pointing along the line (= four-velocity if $\lambda = \tau$).

The components are very similar to the covariant derivative

$$V^\alpha{}_{;\beta} = V^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\gamma\beta} V^\gamma.$$

In fact if we write

$$\frac{dV^\alpha}{d\lambda} = \frac{\partial V^\alpha}{\partial x^\beta}\frac{dx^\beta}{d\lambda} = \frac{\partial V^\alpha}{\partial x^\beta} U^\beta,$$

(a cheat: V^α might only be defined on the line) then we can write

$$\frac{DV^\alpha}{D\lambda} = V^\alpha{}_{;\beta} U^\beta.$$

$DV^\alpha/D\lambda$ is to $dV^\alpha/d\lambda$ as $V^\alpha{}_{;\beta}$ is to $V^\alpha{}_{,\beta}$.

Parallel transport: if a vector $\vec{V}$ is "parallel transported" along a line then

$$\nabla_{\vec{U}} \vec{V} = \frac{d\vec{V}}{d\lambda} = 0,$$

or in component form:

$$\frac{DV^\alpha}{D\lambda} = \frac{dV^\alpha}{d\lambda} + \Gamma^\alpha{}_{\gamma\beta}\frac{dx^\beta}{d\lambda} V^\gamma = 0.$$

Shows how V^α must change for $\vec{V}$ to remain constant.

10.2 Straight lines or "geodesics"

With parallel transport we can extend the idea of "straight" lines to curved spaces:

> Definition: a line is "straight" if it parallel transports its own tangent vector.

In other words straight lines in curved spaces are defined by $\nabla_{\vec{U}} \vec{U} = 0$ or, setting $V^\alpha = U^\alpha = dx^\alpha/d\lambda$

$$\boxed{\frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha{}_{\gamma\beta}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0.}$$

More compactly

$$\boxed{\ddot{x}^\alpha + \Gamma^\alpha{}_{\gamma\beta}\dot{x}^\beta \dot{x}^\gamma = 0,}$$

using the "dot" notation for derivatives wrt λ.

- These are force-free equations of motion
- Extends SR $\vec{A} = d\vec{U}/d\tau = 0$ to GR.

- In GR, gravity is not a force but a distortion of spacetime
- Metric $g_{\alpha\beta} \rightarrow \Gamma^{\gamma}{}_{\alpha\beta} \rightarrow$ particle motion.
- Straight lines are often called geodesics. "Great circles" are geodesics on spheres.

10.2.1 Affine parameters

We could have defined "straight" by $\nabla_{\vec{U}} \vec{U} = k\vec{U}$, i.e. the tangent vector changes by a vector parallel to itself. However in such cases one can always transform to a new parameter, say $\mu = \mu(\lambda)$, such that $\nabla_{\vec{U}'} \vec{U}' = 0$, where $\vec{U}'$ is the new tangent vector. μ is then called an affine parameter. Proper time τ is affine for massive particles.

I will always assume affine parameters.

10.3 Example: motion under a central force

Consider motion under Newtonian gravity

$$\frac{d\vec{V}}{dt} = -\frac{GM}{r^2}\hat{\vec{r}}.$$

In general coordinates the left-hand side is

$$\frac{dV^\alpha}{dt} + \Gamma^\alpha{}_{\beta\gamma} V^\beta V^\gamma.$$

In polar coordinates $\vec{V} = (\dot{r}, \dot{\theta})$.

From last time $\Gamma^r{}_{\theta\theta} = -r$, $\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = 1/r$ with all others $= 0$. Therefore:

$$\frac{dV^r}{dt} + \Gamma^r{}_{\theta\theta} V^\theta V^\theta = -\frac{GM}{r^2},$$

and

$$\frac{dV^\theta}{dt} + \Gamma^\theta{}_{r\theta} V^r V^\theta + \Gamma^\theta{}_{\theta r} V^\theta V^r = 0.$$

These give

$$\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2},$$

and

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} = 0.$$

The second can be integrated to give the well known conservation of angular momentum $r^2\dot{\theta} = h$.

These two equations are the equations of planetary motion which lead to ellipses and Kepler's laws. The point here is how the connection allows one to cope with familiar equations in awkward coordinates. In much of physics such coordinates can be avoided, but not in GR where there is no sidestepping the connection. Note here how the centrifugal term, $r\dot{\theta}^2$, *appears via the connection.*

Lecture 11

Geodesics

Objectives:

- *Variational approach to geodesics*

Reading: *Schutz, 5, 6 & 7; Hobson 5, 7; Rindler 9, 10*

11.1 Extremal Paths

Straight lines are also the shortest. In GR path length is

$$S = \int ds = \int \sqrt{g_{\alpha\beta}\, dx^\alpha\, dx^\beta}.$$

Parameterising by λ:

$$S = \int \sqrt{g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}\, d\lambda.$$

Minimisation of S is a variational problem solvable with the Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) - \frac{\partial L}{\partial x^\alpha} = 0,$$

where $\dot{x}^\alpha = dx^\alpha/d\lambda$ and the Lagrangian is

$$L = \frac{ds}{d\lambda} = \sqrt{g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}}.$$

The square root is inconvenient; consider instead using $L' = (L)^2$ as the Lagrangian. Then the Euler-Lagrange equations would be

$$\frac{d}{d\lambda}\left(2L\frac{\partial L}{\partial \dot{x}^\alpha}\right) - 2L\frac{\partial L}{\partial x^\alpha} = 0.$$

In GR S is actually maximum for straight paths, as a consequence of the minus signs in the metric.

See handout 4

Now if λ satisfies

$$\frac{ds}{d\lambda} = L = \text{constant},$$

then

$$2L \left[\frac{d}{d\lambda} \left(\frac{\partial L}{\partial \dot{x}^\alpha} \right) - \frac{\partial L}{\partial x^\alpha} \right] = 0,$$

so

$$L' = (L)^2 = g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda},$$

leads to the same equations as L provided λ is chosen so that $ds/d\lambda$ is constant (L works for any λ).

The constraint on λ is another way to define affine parameters. Since $ds/d\tau = c$, the speed of light, a constant, proper time is affine.

But remember, proper time cannot be used for photons.

Can show that Euler-Lagrange equations are equivalent to equations of motion derived before, i.e.

$$\ddot{x}^\alpha + \Gamma^\alpha{}_{\gamma\beta} \dot{x}^\beta \dot{x}^\gamma = 0.$$

11.2 Why use the Lagrangian approach?

Application of the Euler-Lagrange equations is often easier than calculating the 40 coefficients of the Levi-Civita connection.

Example 11.1 *Calculate the equations of motion for the Schwarzschild metric*

$$ds^2 = c^2 \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \frac{dr^2}{1 - 2GM/c^2 r} - r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right),$$

using the Euler-Lagrange approach.

Answer 11.1 *Setting* $dt \to \dot{t}$, $dr \to \dot{r}$, $d\theta \to \dot{\theta}$ *and* $d\phi \to \dot{\phi}$ *in* ds^2, *the Lagrangian is given by*

$$L = c^2 \left(1 - \frac{2GM}{c^2 r}\right) \dot{t}^2 - \frac{\dot{r}^2}{1 - 2GM/c^2 r} - r^2 \left(\dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2\right).$$

Consider, say, the θ *component of the E-L equations:*

$$\frac{d}{d\lambda} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0.$$

This gives

$$\frac{d}{d\lambda} \left(-2r^2 \dot{\theta} \right) + 2r^2 \sin\theta \cos\theta \, \dot{\phi}^2 = 0,$$

much more directly than the connection approach.

11.3 Conserved quantities

If L does not depend explicitly on a coordinate x^α say, then $\partial L/\partial x^\alpha = 0$, and so the E-L equations show that

$$\frac{\partial L}{\partial \dot{x}^\alpha} = 2g_{\alpha\beta}\dot{x}^\beta = 2\dot{x}_\alpha = \text{constant.}$$

In other words the covariant component of the corresponding velocity is conserved.

e.g. The metric of the example does not depend upon ϕ so

$$r^2 \sin^2(\theta)\,\dot{\phi} = \text{constant.}$$

When motion confined to equatorial plane $\theta = \pi/2$, $r^2\dot{\phi} = h$, a constant: GR equivalent of angular momentum conservation.

11.4 Slow motion in a weak field

Consider equations of motion at slow speeds in weak fields. Mathematically $\dot{x}^i \to 0$ for $i = 1$, 2 or 3, and $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ where $|h_{\alpha\beta}| \ll 1$. The equations of motion

$$\ddot{x}^\alpha + \Gamma^\alpha{}_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma = 0,$$

reduce to

$$\ddot{x}^\alpha + \Gamma^\alpha{}_{00}\dot{x}^0\dot{x}^0 = 0.$$

The time "velocities" $\dot{x}^0$ are never negligible, and in fact for $\lambda = \tau$, are $d(ct)/d\tau \approx c$.

From the Levi-Civita equation, retaining terms to first order in h

$$\begin{aligned}\Gamma^\alpha{}_{00} &= \frac{1}{2}g^{\alpha\beta}(g_{\beta 0,0} + g_{0\beta,0} - g_{00,\beta}),\\ &= \frac{1}{2}\eta^{\alpha\beta}(h_{\beta 0,0} + h_{0\beta,0} - h_{00,\beta}).\end{aligned}$$

If the metric is stationary, all time derivatives ("$,0$" terms) are zero, and so

$$\Gamma^0{}_{00} = \frac{1}{2}(h_{00,0} + h_{00,0} - h_{00,0}) = 0,$$

since all time derivatives are zero. Therefore $\ddot{x}^0 = 0$ or $\dot{x}^0$ is constant. The spatial components become

$$\Gamma^i{}_{00} = -\frac{1}{2}(h_{i0,0} + h_{0i,0} - h_{00,i}) = \frac{1}{2}h_{00,i},$$

($\eta^{ii} = -1$ for each i, stationary metric) giving

$$\ddot{x}^i = -\frac{1}{2} h_{00,i} \dot{x}^0 \dot{x}^0.$$

Since $\dot{x}^0 = c dt/d\tau$ is constant, we finally obtain

$$\frac{d^2 x^i}{dt^2} = -\frac{1}{2} c^2 h_{00,i},$$

or

$$\ddot{\mathbf{r}} = -\frac{1}{2} c^2 \nabla h_{00}.$$

(dots now derivatives wrt t not τ). What is h_{00}? Consider clock at rest then

$$c^2\, d\tau^2 = g_{00} c^2\, dt^2,$$

or

$$d\tau = \sqrt{1 + h_{00}} = \left(1 + \frac{h_{00}}{2}\right) dt.$$

But equivalence principle $\implies$

$$d\tau = \left(1 + \frac{\phi}{c^2}\right) dt,$$

so

$$h_{00} = \frac{2\phi}{c^2},$$

where ϕ is Newtonian gravitational potential. Therefore

$$\ddot{\mathbf{r}} = -\nabla \phi,$$

the equation of motion in Newtonian gravity!

ϕ is the Newtonian equivalent to the g_{00} component of the metric. None of the other metric components are represented in Newton's theory.

This finally completes the loop of establishing that motion in a curved space-time can give rise to what until now we have called the force of gravity. On Earth $h_{00} \sim 10^{-9}$. It is amazing that so tiny a wrinkle of spacetime leads to the phenomenon of gravity. We must next see how mass determines the metric.

Lecture 12

Curvature

Objectives:

- *Curvature and geodesic deviation*

Reading: *Schutz, 6; Hobson 7; Rindler 10.*

12.1 Local inertial coordinates

The metric determines particle motion, and Newton's Law of Gravity, $\nabla^2\phi = 4\pi G\rho$, suggests that mass must fix the metric. Thus we seek a tensor built from the metric and/or its derivatives that can substitute for $\nabla^2\phi$ in Newton's theory.

$g_{\alpha\beta}$ alone is no good because coordinates can always be found such that $g_{\alpha\beta} = \eta_{\alpha\beta}$, the Minkowski mertic. This clearly cannot simultaneously describe situations with and without mass.

Proof: there are 10 independent coefficients of $g_{\alpha\beta}$ but 16 degrees of freedom in the transformation matrix, $\partial x^\beta / \partial x^{\alpha'}$.

The first derivatives $\partial g_{\alpha\beta}/\partial x^\gamma = g_{\alpha\beta,\gamma}$ are not enough either, because it can be shown that coordinates can always be found in which

$$g_{\alpha\beta,\gamma} = 0.$$

In these coordinates, the Levi-Civita equation implies

$$\Gamma^\alpha{}_{\beta\gamma} = 0,$$

so that $A^\alpha = dU^\alpha/d\tau = 0$. These are locally inertial or geodesic coordinates, the freely-falling frames of the equivalence principle.

Corollary: in an inertial frame, covariant derivative $\rightarrow$ ordinary partial derivative $\Longrightarrow$

$$\boxed{g_{\alpha\beta;\gamma} = g_{\alpha\beta,\gamma} = 0,}$$

$g_{\alpha\beta;\gamma} = 0$ is tensorial, so the metric is covariantly constant, $\nabla \mathbf{g} = 0$.

Conclusion: we need a tensor involving at least second derivatives of the metric, as suggested by $\nabla^2\phi$ and $g_{00} \approx 1 + 2\phi/c^2$.

12.2 Curvature tensor

Consider the expression

$$\nabla_\gamma \nabla_\beta V_\alpha = V_{\alpha;\beta\gamma},$$

where $\tilde{V}$ is an arbitrary one-form. This is a tensor (derivatives are covariant) which contains second derivatives of the metric. Expanding the covariant derivative with respect to γ:

$$\begin{aligned} V_{\alpha;\beta\gamma} &= [V_{\alpha;\beta}]_{;\gamma}\,, \\ &= V_{\alpha;\beta,\gamma} - \Gamma^\sigma{}_{\alpha\gamma} V_{\sigma;\beta} - \Gamma^\sigma{}_{\beta\gamma} V_{\alpha;\sigma}. \end{aligned}$$

Each of the three covariant derivatives, $V_{\alpha;\beta}$ etc, can be expanded similarly and one ends up with an expression of the form

$$V_{\alpha;\beta\gamma} = [\ldots]V_{\mu,\beta\gamma} + [\ldots]V_{\rho,\sigma} + [\ldots]V_\rho.$$

The terms in brackets involve second derivatives of g. Unfortunately although the sum is a tensor, we cannot assert that the individual terms are tensors: we need just one term involving V_ρ alone. See handout 5

If instead we consider the tensor $V_{\alpha;\beta\gamma} - V_{\alpha;\gamma\beta}$, the derivatives in V cancel and we find See handout 5

$$\boxed{V_{\alpha;\beta\gamma} - V_{\alpha;\gamma\beta} = [\nabla_\gamma, \nabla_\beta]\, V_\alpha = R^\rho{}_{\alpha\beta\gamma} V_\rho.}$$

where $R^\rho{}_{\alpha\beta\gamma}$ is the Riemann curvature tensor and is given by Do not try to memorise this!!

$$\boxed{R^\rho{}_{\alpha\beta\gamma} = \Gamma^\rho{}_{\alpha\gamma,\beta} - \Gamma^\rho{}_{\alpha\beta,\gamma} + \Gamma^\sigma{}_{\alpha\gamma}\Gamma^\rho{}_{\sigma\beta} - \Gamma^\sigma{}_{\alpha\beta}\Gamma^\rho{}_{\sigma\gamma}.}$$

In flat spacetime, one can find a coordinate system in which the connection and its derivatives $= 0$, and so

$$R^\rho{}_{\alpha\beta\gamma} = 0.$$

i.e. the Riemann tensor vanishes in flat spacetime. (i.e. covariant differentiation is commutative in flat space.)

12.3 Understanding the curvature tensor

Pictorially the relation

$$V_{\alpha;\beta\gamma} - V_{\alpha;\gamma\beta} = R^{\rho}{}_{\alpha\beta\gamma} V_{\rho},$$

corresponds to the following:

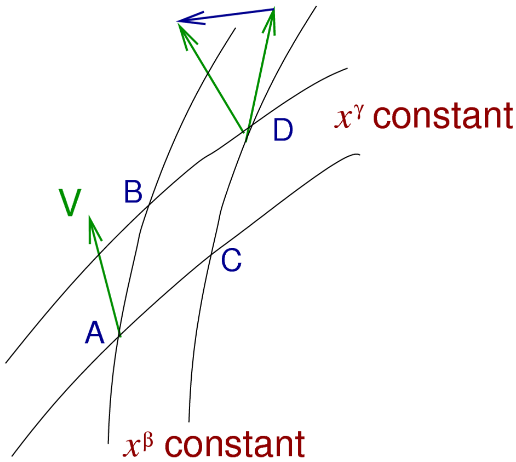

Figure: Vector parallel transported two ways around the same loop does not match up at the end if there is curvature

Vector $\vec{V}$ is first parallel transported $A \to C \to D$, associated with $V^{\alpha}{}_{;\beta\gamma}$. Then the same vector is taken $A \to B \to D$, associated with $V^{\alpha}{}_{;\gamma\beta}$. Curvature causes the vectors at D to differ.

Related to this, a vector parallel-transported around a loop in a curved space changes, e.g.

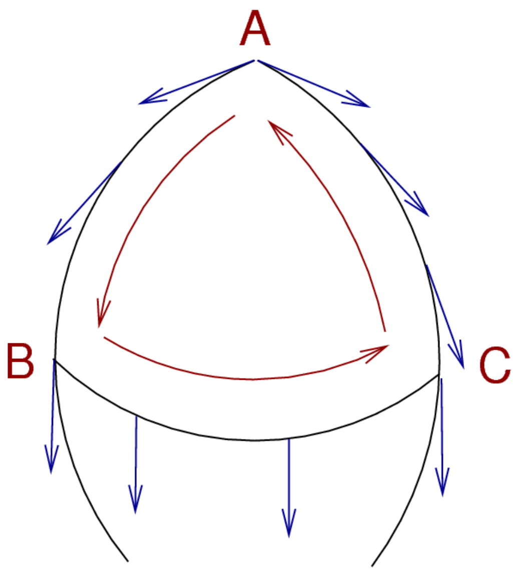

Figure: Vector parallel transported on a sphere A to B to C to A has changed by the time it gets back to A.

12.4 Geodesic Deviation

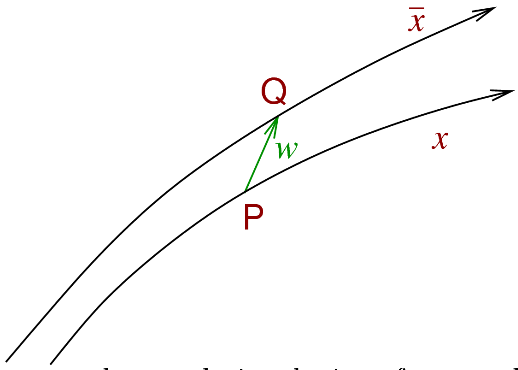

Figure: Two nearby geodesics deviate from each other because of curvature

Consider the relative distance $\vec{w}$ between two nearby particles at P and Q undergoing geodesic motion (free-fall). Can show that

$$\boxed{\frac{D^2 w^\alpha}{D\lambda^2} + R^\alpha{}_{\gamma\beta\delta}\,\dot{x}^\gamma \dot{x}^\delta w^\beta = 0,}$$

where $\dot{x}^\gamma = dx^\gamma/d\lambda$ etc. This is a tensor equation, the equation of geodesic deviation. Here the capital D's indicate 'absolute' or 'total' derivatives, i.e. derivatives that allow for variations in components caused purely by curved coordinates, so that we expect

$$\frac{D^2 w^\alpha}{D\lambda^2} = 0,$$

in the absence of gravity.

The second term therefore represents the effect of gravity that is not removed by free-fall, i.e. it is the tidal acceleration. In Newtonian physics tides are caused by a variation in the gravitational field, $\nabla\mathbf{g}$, and since $\mathbf{g} = -\nabla\phi$, tides are related to $\nabla^2\phi$. This is another indication of the connection between curvature and the left-hand side of $\nabla^2\phi = 4\pi G\rho$.

This is the quantitative version of the notion from chapter 7 of two particles falling towards a gravitating mass moving on initially parallel-paths in space-time which remain straight and yet ultimately meet.

Lecture 13

Einstein's field equations

Objectives:

- *The GR field equations*

Reading: *Schutz, 6; Hobson 7; Rindler 10.*

13.1 Symmetries of the curvature tensor

With 4 indices, the curvature tensor has a forbidding 256 components. Luckily several symmetries reduce these substantially. These are best seen in fully covariant form:

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\rho} R^{\rho}{}_{\beta\gamma\delta},$$

for which symmetries such as

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta},$$

and

$$R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma}.$$

can be proved. These relations reduce the number of independent components to 20. Handout 6

These symmetries also mean that there is only one independent contraction

$$R_{\alpha\beta} = R^{\rho}{}_{\alpha\beta\rho},$$

because others are either zero, e.g.

$$R^{\rho}{}_{\rho\alpha\beta} = g^{\rho\sigma} R_{\sigma\rho\alpha\beta} = 0,$$

or the same to a factor of ± 1. $R_{\alpha\beta}$ is called the Ricci tensor, while its contraction

$$R = g^{\alpha\beta} R_{\alpha\beta},$$

is called the Ricci scalar.

NB Signs vary between books. I follow Hobson et al and Rindler.

13.2 The field equations

We seek a relativistic version of the Newtonian equation

$$\nabla^2\phi = 4\pi G\rho.$$

The relativistic analogue of the density ρ is the stress–energy tensor $T^{\alpha\beta}$.

ϕ is closely related to the metric, and ∇^2 suggests that we look for some tensor involving the second derivatives of the metric, $g_{\alpha\beta,\gamma\delta}$, which should be a $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ tensor like $T^{\alpha\beta}$.

The contravariant form of the Ricci tensor satisfies these conditions, suggesting the following:

$$R^{\alpha\beta} = kT^{\alpha\beta},$$

where k is some constant. (NB both $R^{\alpha\beta}$ and $T^{\alpha\beta}$ are symmetric.)

However, in SR $T^{\alpha\beta}$ satisfies the conservation equations ${T^{\alpha\beta}}_{,\alpha} = 0$ which in GR become

$${T^{\alpha\beta}}_{;\alpha} = 0,$$

whereas it turns out that

$${R^{\alpha\beta}}_{;\alpha} = \frac{1}{2}R_{,\alpha}g^{\alpha\beta} \neq 0,$$

where R is the Ricci scalar. Therefore $R^{\alpha\beta} = kT^{\alpha\beta}$ cannot be right. Handout 6

Fix by defining a new tensor, the <u>Einstein tensor</u>

$$G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta},$$

because then

$${G^{\alpha\beta}}_{;\alpha} = \left(R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta}\right)_{;\alpha} = {R^{\alpha\beta}}_{;\alpha} - \frac{1}{2}R_{;\alpha}g^{\alpha\beta} - \frac{1}{2}R{g^{\alpha\beta}}_{;\alpha} = 0,$$

since $\nabla\mathbf{g} = 0$ and $R_{;\alpha} = R_{,\alpha}$. Therefore we modify the equations to

$$\boxed{R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta} = kT^{\alpha\beta}.}$$

These are <u>Einstein's field equations</u>.

13.3 The Newtonian limit

The equations must reduce to $\nabla^2\phi = 4\pi G\rho$ in the case of slow motion in weak fields. To show this, it is easier to work with an alternate form: contracting the field equations with $g_{\alpha\beta}$ then

$$g_{\alpha\beta}R^{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}g^{\alpha\beta} = kg_{\alpha\beta}T^{\alpha\beta},$$

and remembering the definition of R and defining $T = g_{\alpha\beta}T^{\alpha\beta}$,

$$R - \frac{1}{2}\delta^\alpha_\alpha R = -R = kT,$$

since $\delta^\alpha_\alpha = 4$. Therefore

$$\boxed{R^{\alpha\beta} = k\left(T^{\alpha\beta} - \frac{1}{2}Tg^{\alpha\beta}\right).}$$

Easier still is the covariant form:

$$R_{\alpha\beta} = k\left(T_{\alpha\beta} - \frac{1}{2}Tg_{\alpha\beta}\right).$$

The stress–energy tensor is

$$T_{\alpha\beta} = \left(\rho + \frac{p}{c^2}\right)U_\alpha U_\beta - pg_{\alpha\beta}.$$

In the Newtonian case, $p/c^2 \ll \rho$, and so

$$T_{\alpha\beta} \approx \rho U_\alpha U_\beta.$$

Therefore

$$T = g^{\alpha\beta}T_{\alpha\beta} = \rho g^{\alpha\beta}U_\alpha U_\beta = \rho c^2.$$

Weak fields imply $g_{\alpha\beta} \approx \eta_{\alpha\beta}$, so $g_{00} \approx 1$. For slow motion, $U^i \ll U^0 \approx c$, and so $U_0 = g_{0\alpha}U^\alpha \approx g_{00}U^0 \approx c$ too. Thus

$$T_{00} \approx \rho c^2,$$

is the only significant component.

The 00 cpt of $R_{\alpha\beta}$ is:

$$R_{00} = \Gamma^\rho{}_{0\rho,0} - \Gamma^\rho{}_{00,\rho} + \Gamma^\sigma{}_{0\rho}\Gamma^\rho{}_{\sigma 0} - \Gamma^\sigma{}_{00}\Gamma^\rho{}_{\sigma\rho}.$$

All Γ are small, so the last two terms are negligible. Then assuming time-independence,

$$R_{00} \approx -\Gamma^i{}_{00,i}.$$

But, from the lecture on geodesics (chapter 11),

$$\Gamma^i{}_{00} = \frac{\phi_{,i}}{c^2}.$$

Thus

$$R_{00} \approx -\frac{1}{c^2}\phi_{,ii} = -\frac{1}{c^2}\frac{\partial^2\phi}{\partial x^i \partial x^i} = -\frac{1}{c^2}\nabla^2\phi.$$

No longer balancing up/down indices since we are referring to spatial components only in nearly-flat space-time.

Finally, substituting in the field equations

$$-\frac{1}{c^2}\nabla^2\phi = k\left(\rho c^2 - \frac{1}{2}\rho c^2\right),$$

or

$$\nabla^2\phi = -\frac{kc^4}{2}\rho.$$

Therefore if $k = -8\pi G/c^4$, we get the Newtonian equation as required, and the field equations become

$$\boxed{R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta} = -\frac{8\pi G}{c^4}T^{\alpha\beta}.}$$

Key points:

- The field equations are second order, non-linear differential equations for the metric
- 10 independent equations replace $\nabla^2\phi = 4\pi G\rho$
- By design they satisfy the energy-momentum conservation relations ${T^{\alpha\beta}}_{;\alpha} = 0$
- The constant $8\pi G/c^4$ gives the correct Newtonian limit
- Although derived from strong theoretical arguments, like any physical theory, they can only be tested by experiment.

Lecture 14

Schwarzschild geometry

Objectives:

- *Schwarzschild's solution*

Reading: *Schutz, 10; Hobson 9; Rindler 11; Foster & Nightingale 3.*

14.1 Isotropic metrics

It is hard to solve the field equations. Symmetry arguments are essential. The first such solution to the field equations was derived by Schwarzschild in 1916 for spherical symmetry.

Consider first the Minkowski interval

$$ds^2 = c^2\,dt^2 - dr^2 - r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right).$$

The term in brackets expresses spherical symmetry or isotropy (no preference for any direction). Any spherically symmetric metric must have a term of this form. Thus a general isotropic metric can be written

$$ds^2 = A\,dt^2 - B\,dt\,dr - C\,dr^2 - D\left(d\theta^2 + \sin^2\theta\,d\phi^2\right).$$

- Expect symmetry under $\phi \rightarrow -\phi$, $\theta \rightarrow \pi - \theta$ so no cross terms with $dr\,d\theta$ or $d\phi\,dt$.
- A, B, C and D cannot depend on θ or ϕ otherwise isotropy is broken $\Longrightarrow$ functions of r and t only.

We can <u>define</u> a new radial coordinate r' such that $(r')^2 = D$, and so the metric becomes

$$ds^2 = A'\,dt^2 - B'\,dt\,dr' - C'\,(dr')^2 - (r')^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right).$$

This metric is still general.

Dropping the primes, with this radial coordinate, the area of a sphere is still $4\pi r^2$, but r is not necessarily the ruler distance from the origin.

Finally we can transform the time coordinate using

$$dt = f\, dt' + g\, dr,$$

choosing f and g such that dt is an exact differential and so that the cross terms in $dr\, dt'$ cancel. We are left with

Dropping primes

$$\boxed{ds^2 = A(r,t)\, dt^2 - B(r,t)\, dr^2 - r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right).}$$

as the general form of an isotropic metric.

14.2 Schwarzschild metric

We specialise further by looking for time-independent metrics, i.e.

$$ds^2 = A(r)\, dt^2 - B(r)\, dr^2 - r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right).$$

This is also static as it is invariant under the transform $t \to -t$.

We want to find the metric around a star such as the Sun, i.e. in empty space where $T_{\alpha\beta} = 0$ and $T = T^\alpha_\alpha = 0 \implies R = 0$, so the field equations

$$\left(R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}\right) = -\frac{8\pi G}{c^4} T_{\alpha\beta},$$

reduce to

$$R_{\alpha\beta} = 0.$$

$R_{\alpha\beta}$ comes from

$$R_{\alpha\beta} = \Gamma^\rho{}_{\alpha\beta,\rho} - \Gamma^\rho{}_{\alpha\rho,\beta} + \Gamma^\sigma{}_{\alpha\beta}\Gamma^\rho{}_{\sigma\rho} - \Gamma^\sigma{}_{\alpha\rho}\Gamma^\rho{}_{\sigma\beta},$$

while

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(g_{\delta\gamma,\beta} + g_{\beta\delta,\gamma} - g_{\beta\gamma,\delta}\right).$$

Unfortunately there are no more short-cuts from this point. Work out Γ then R. Much algebra leads to coupled, ordinary differential equations for A and B (e.g. Hobson et al p200) and one finds

See Q8, problem sheet 4, Q5, sheet 5

$$\begin{aligned} A(r) &= \alpha \left(1 + \frac{k}{r}\right), \\ B(r) &= \left(1 + \frac{k}{r}\right)^{-1}, \end{aligned}$$

α and k constants.

In weak fields we know that

$$A(r) \to c^2 \left(1 + \frac{2\phi}{c^2}\right),$$

so $\alpha = c^2$ and $k = -2GM/c^2$. We arrive at the Schwarzschild metric:

$$\boxed{ds^2 = c^2 \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right).}$$

This applies outside a spherically-symmetric object, e.g. for motions of the planets but not inside the Sun.

Schwarzschild's solution is important as the first exact solution of the field equations.

14.3 Birkhoff's theorem

If one does not impose time-independence, i.e. $A = A(r,t)$, $B = B(r,t)$, and solves $R_{\alpha\beta} = 0$, one still finds Schwarzschild's solution (Birkhoff 1923), i.e.

> The geometry outside a spherically symmetric distribution of matter is the Schwarzschild geometry.

This means spherically symmetric explosions cannot emitt gravitational waves.

It also means that spacetime inside a hollow spherical shell is flat since it must be Schwarzschild-like but have $M = 0$. Flat implies no gravity, the GR equivalent of Newton's "iron sphere" theorem.

Used in semi-Newtonian justifications of the Friedmann equations.

14.4 Schwarzschild radius

The Schwarzschild metric has a singularity at

$$\boxed{r = R_S = \frac{2GM}{c^2} = 2.9 \frac{M}{\mathrm{M}_\odot}\,\mathrm{km}.}$$

Usually this is irrelevant, because the Schwarzschild radius lies well inside typical objects where the metric does not apply, e.g. for the Sun $R_S \ll \mathrm{R}_\odot = 7 \times 10^5$ km, for Earth $R_S \approx 1$ cm.

However, it is easy to conceive circumstances where objects have $R < R_S$, e.g consider the Galaxy as 10^{11} Sun-like stars. Then

$$R_S = 2.9 \times 10^{11}\,\mathrm{km},$$

$\sim 50\times$ size of Solar system. Mean distance between N stars in a sphere radius R_S

$$d = \left(\frac{4\pi R_S^3}{3N}\right)^{1/3} = 1.00 \times 10^8\,\mathrm{km}.$$

Comparing with $\mathrm{R}_\odot = 7 \times 10^5\,\mathrm{km}$, the stars have plenty of space: do not require extreme density.

Finally, as a hint of things to come, consider the interval for $r < R_S$. Then $g_{tt} = c^2\,(1 - R_S/r) < 0$ and $g_{rr} = -(1 - R_s/r)^{-1} > 0$. Massive particles must have $ds^2 > 0$, but, ignoring θ and ϕ,

$$ds^2 = c^2\,d\tau^2 = g_{tt}\,dt^2 + g_{rr}\,dr^2 > 0.$$

Given that $g_{tt} < 0$ and $g_{rr} > 0$, we must have $dr \neq 0$ for $r < R_S$ to give $ds^2 > 0$. The passing of proper time therefore requires a change in radial coordinate; the future "points" inwards. This leads to a collapse to a singularity at $r = 0$. There is no such thing as a stationary observer for $r < R_S$.

Lecture 15

Schwarzschild equations of motion

Objectives:

- *Planetary motion, start.*

Reading: *Schutz, 11; Hobson 9; Rindler 11.*

15.1 Equations of motion

Writing $\mu = GM/c^2$, the Schwarzschild metric becomes

$$ds^2 = c^2 \left(1 - \frac{2\mu}{r}\right) dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right),$$

and the corresponding Lagrangian is

$$L = c^2 \left(1 - \frac{2\mu}{r}\right) \dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} \dot{r}^2 - r^2 \left(\dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2\right).$$

There is no explicit dependence on either t or ϕ, and thus $\partial L/\partial \dot{t}$ and $\partial L/\partial \dot{\phi}$ are constants of motion, i.e

$$\begin{aligned} \left(1 - \frac{2\mu}{r}\right) \dot{t} &= k, \\ r^2 \sin^2\theta \, \dot{\phi} &= h, \end{aligned}$$

where k and h are constants. h is the GR equivalent of angular momentum per unit mass.

For k, recall that for "ignorable coordinates" such at t and ϕ, the corresponding covariant velocity is conserved , i.e.

$$\dot{x}_0 = g_{0\beta}\dot{x}^\beta = g_{00}\dot{x}^0 = \text{constant},$$

where the third term follows from diagonal metric. Now $x^0 = ct$, while $g_{00} = 1 - 2\mu/r$, so

$$\dot{x}_0 = \left(1 - \frac{2\mu}{r}\right) c\dot{t} = kc.$$

Now $p_0 = m\dot{x}_0$, where p_0 is the time component of the four-momentum, and in flat spacetime $p_0 = E/c$ where E is the energy, so

$$E = p_0 c = \dot{x}_0 mc = kmc^2,$$

is the total energy for motion in a Schwarzschild metric.

NB k can be < 1, because in Newtonian terms it contains potential energy as well as kinetic and rest mass energy.

For the r component we have

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0,$$

which gives

$$\frac{d}{d\lambda}\left(-\left(1 - \frac{2\mu}{r}\right)^{-1} 2\dot{r}\right) - \left(\frac{2\mu c^2}{r^2}\dot{t}^2 + \left(1 - \frac{2\mu}{r}\right)^{-2} \frac{2\mu}{r^2}\dot{r}^2 - 2r\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right)\right).$$

while the θ component leads to:

$$\frac{d}{d\lambda}\left(-2r\dot{\theta}\right) - \left(-2r^2\sin\theta\cos\theta\,\dot{\phi}^2\right) = 0.$$

The last equation is satisfied for $\theta = \pi/2$, i.e. motion in the equatorial plane. By symmetry, we need not consider any other case, leaving

$$\begin{aligned}\left(1 - \frac{2\mu}{r}\right)\dot{t} &= k,\\ \left(1 - \frac{2\mu}{r}\right)^{-1}\ddot{r} + \frac{\mu c^2}{r^2}\dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-2}\frac{\mu}{r^2}\dot{r}^2 - r\dot{\phi}^2 &= 0,\\ r^2\dot{\phi} &= h.\end{aligned}$$

For circular motion, $\dot{r} = \ddot{r} = 0$, the second equation reduces to

$$\frac{\mu c^2}{r^2}\dot{t}^2 = r\dot{\phi}^2,$$

and defining $\omega_\phi = d\phi/dt$ and remembering $\mu = GM/c^2$, we get

$$\omega_\phi^2 = \frac{GM}{r^3},$$

Kepler's third law! ...somewhat luckily because of the choice of r and t.

15.2 An easier approach

Rather than use the radial equation above, it is easier to use another constant of geodesic motion:

$$\vec{U} \cdot \vec{U} = g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = \text{constant}.$$

This is effectively a first integral which comes from the affine constraint, or, equivalently, from $\nabla_{\vec{U}}\vec{U} = 0$. It side-steps the $\ddot{r}$ term.

More specifically we have

$$g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = c^2,$$

for massive particles with $\lambda = \tau$, and

$$g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = 0,$$

for photons.

15.3 Motion of massive particles

The equations to be solved in this case are thus

$$\begin{aligned}
\left(1 - \frac{2\mu}{r}\right)\dot{t} &= k, \\
c^2\left(1 - \frac{2\mu}{r}\right)\dot{t}^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 &= c^2, \\
r^2\dot{\phi} &= h.
\end{aligned}$$

Substituting for $\dot{t}$ and $\dot{\phi}$ in the second equation and multiplying by $-(1 - 2\mu/r)$ gives

$$\boxed{\dot{r}^2 + \frac{h^2}{r^2}\left(1 - \frac{2\mu}{r}\right) - \frac{2\mu c^2}{r} = c^2\left(k^2 - 1\right).}$$

This has the form of an energy equation with a "kinetic energy" term, $\dot{r}^2$ plus a function of r, "potential energy" equalling a constant.

Thus the motion in the radial coordinate is exactly equivalent to a particle moving in an effective potential $V(r)$ where

$$V(r) = \frac{h^2}{2r^2}\left(1 - \frac{2\mu}{r}\right) - \frac{\mu c^2}{r},$$

or, setting $\mu = GM/c^2$,

$$V(r) = \frac{h^2}{2r^2}\left(1 - \frac{2GM}{c^2 r}\right) - \frac{GM}{r}.$$

One can learn much about Schwarzschild orbits from this potential.

The equivalent in Newtonian mechanics is easy to derive:

$$\dot{r}^2 + r^2\dot{\phi}^2 - \frac{2GM}{r} = \frac{2E}{m},$$

and $r^2\dot{\phi} = h$. Thus

$$\dot{r}^2 + \frac{h^2}{r^2} - \frac{2GM}{r} = \frac{2E}{m},$$

so

$$V_N(r) = \frac{h^2}{2r^2} - \frac{GM}{r}.$$

GR introduces an extra term in $1/r^3$ in addition to the Newtionian $1/r$ gravitational potential and $1/r^2$ "centrifugal barrier" terms.

15.4 Schwarzschild orbits

Three movies of orbits in Schwarzschild geometry were shown in the lecture.

Movies illustrate the following key differences between GR and Newtonian predictions:

- Apsidal precession of elliptical orbits
- Instability of close-in circular orbits
- Capture orbits

Lecture 16

Schwarzschild orbits

Objectives:

- *Planetary motion*

Reading: *Schutz, 11; Hobson 9; Rindler 11*

16.1 Newtonian orbits

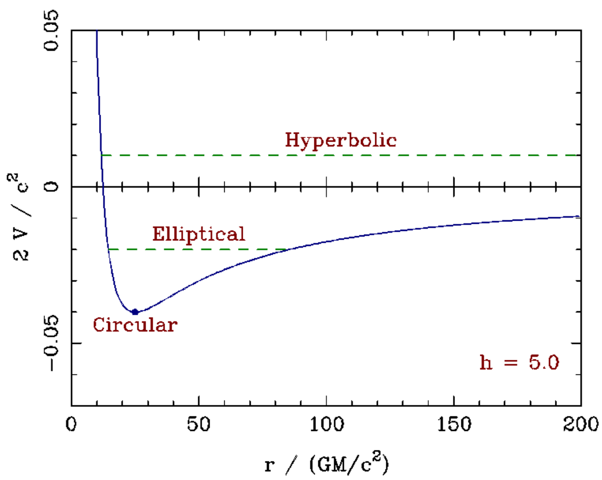

Figure: Newtonian effective potential: centrifugal barrier always wins

- Centrifugal barrier always dominates as $r \rightarrow 0$
- 2 types of orbits: unbound, hyperbolic $E > 0$; bound, elliptical $E < 0$.

- Circular: $\dot{r} = 0$, $r = r_C$ such that $\ddot{r} = 0 \implies dV/dr = V'(r) = 0$.
- Newtonian elliptical orbits do not precess.

To see last point, expand potential around $r = r_C$:

$$V(r) \approx V(r_c) + \frac{1}{2}V''(r_C)(r - r_C)^2.$$

cf potential/unit mass of a spring $kx^2/2m$, then r must oscillate with angular frequency ("epicyclic frequency")

$$\omega_r^2 = V''(r_C).$$

Given the Newtonian effective potential

$$V(r) = \frac{h^2}{2r^2} - \frac{GM}{r},$$

so

$$V'(r) = \frac{-h^2}{r^3} + \frac{GM}{r^2}.$$

$V'(r_C) = 0 \implies h^2 = GMr_C$, therefore

$$V''(r_C) = \frac{3h^2}{r_C^4} - \frac{2GM}{r_C^3} = \frac{GM}{r_C^3}.$$

However, $\omega_\phi^2 = GM/r_C^3$, thus $\omega_r = \omega_\phi$. $\implies$ always reach minimum r at same ϕ, so no precession.

16.2 Schwarzschild orbits

Case 1. Large angular momentum h

Reminder: $V(r) = \frac{h^2}{2r^2}\left(1 - \frac{2\mu}{r}\right) - \frac{\mu c^2}{r}$.

Units of h on plots are μc.

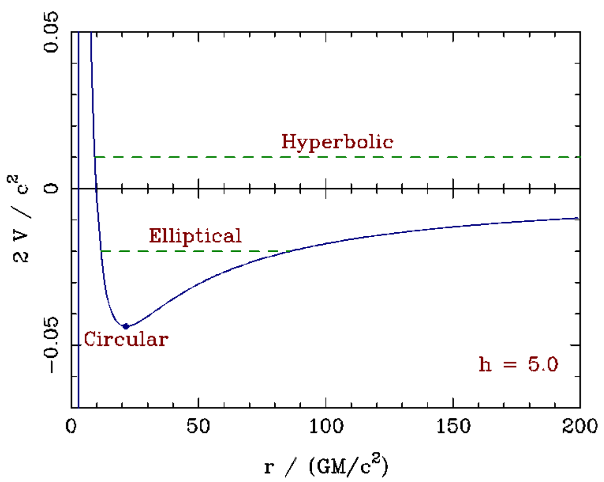

Figure: Schwarzschild effective potential for a large values of h

- Essentially Newtonian behaviour as small r is inaccessible.
- This case applies to the planets. e.g. for Earth $h \approx 10^4 \mu c$.

Case 2. Intermediate angular momentum h

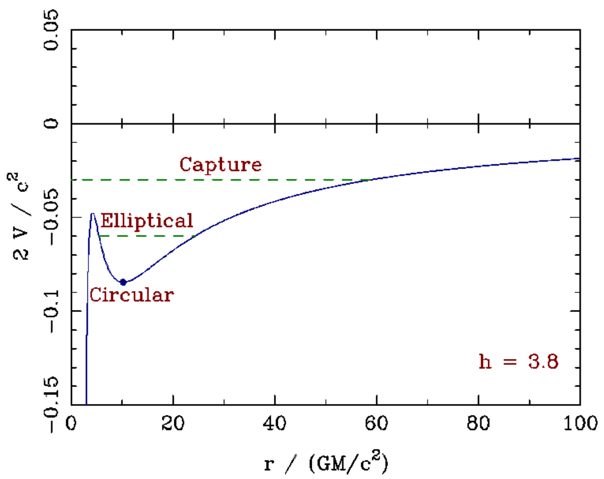

Figure: Schwarzschild effective potential for an intermediate value of h

- Bound near-elliptical and circular orbits still exist
- Qualitatively different capture orbits possible.

Case 3. Low angular momentum h

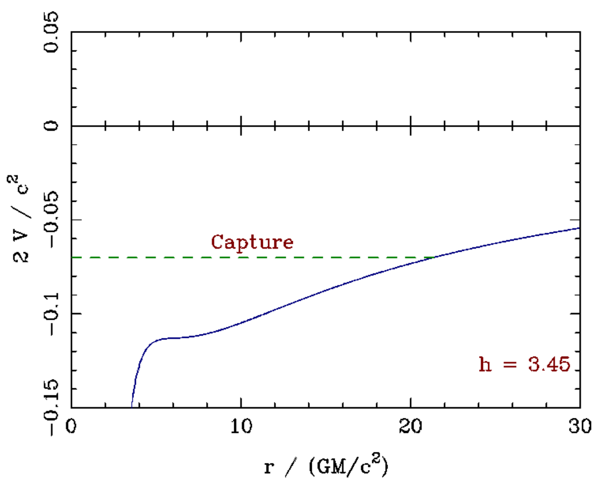

Figure: Schwarzschild effective potential for a low value of h

- No bound orbits.

16.2.1 Instability of circular orbits

The Schwarzschild effective potential is

$$V(r) = \frac{h^2}{2r^2}\left(1 - \frac{2\mu}{r}\right) - \frac{\mu c^2}{r}.$$

At the radius of circular orbits, $dV(r)/dr = V'(r) = 0 \implies$

$$V'(r) = -\frac{h^2}{r^3} + \frac{3h^2\mu}{r^4} + \frac{\mu c^2}{r^2} = 0,$$

or

$$\mu c^2 r^2 - h^2 r + 3h^2\mu = 0,$$

so

$$r_C = \frac{h^2 \pm \sqrt{h^4 - 12h^2\mu^2c^2}}{2\mu c^2}.$$

The smaller root is a maximum of V and unstable. The larger root is stable while $h^2 > 12\mu^2c^2$, but once $h^2 \leq 12\mu^2c^2$ there are no more stable circular orbits. At this point

$$r_C = \frac{h^2}{2\mu c^2} = 6\mu = \frac{6GM}{c^2} = 3R_S.$$

In accretion discs around non-rotating black-holes no more energy is available from within this radius. Calculate energy lost using $E = kmc^2$.

Since $\dot{r} = 0$, $r = 6\mu$ and $h^2 = 12\mu^2c^2$:

$$\begin{aligned} c^2(k_C^2 - 1) &= \frac{h^2}{r^2}\left(1 - \frac{2\mu}{r}\right) - \frac{2\mu c^2}{r}, \\ &= \frac{12\mu^2c^2}{36\mu^2}\left(1 - \frac{2}{6}\right) - \frac{2c^2}{6}, \\ &= -\frac{1}{9}c^2. \end{aligned}$$

Thus $k_C^2 = 8/9$. A mass dropped from rest at $r = \infty$ starts with $k = 1$, and thus $1 - k_C = 5.7\,\%$ of the rest mass must be lost to radiation. Compare with $\sim 0.7\,\%$ H $\rightarrow$ He fusion.

cf "Newtonian" value of $GM/6R_S = 1/12 = 8.3\%$.

Accretion power from black-holes is thus a conservative hypothesis in many cases as it requires much less fuel than fusion, e.g. 1 star per week rather than 7 or 8. Rotating black-holes can be more efficient still, with a maximum of 42% (Kerr metrics). In realistic cases it is thought that about 30% efficiency is possible.

Lecture 17

Precession and Photon orbits

Objectives:

- *Precession of perihelion*
- *Start on orbits of photons*

Reading: *Schutz, 10 & 11; Hobson 9 & 10; Rindler 11.*

17.0.2 Precession in the Schwarzschild geometry

As for Newton, oscillations in r occur at $\omega_r^2 = V''(r_c)$ but now, setting $\mu = GM/c^2$,

$$V(r) = \frac{h^2}{2r^2}\left(1 - \frac{2\mu}{r}\right) - \frac{\mu c^2}{r} = h^2\left(\frac{1}{2r^2} - \frac{\mu}{r^3}\right) - \frac{\mu c^2}{r}.$$

First obtain a condition on h for circular orbits of radius r from $V'(r) = 0$:

$$V'(r) = h^2\left(-\frac{1}{r^3} + \frac{3\mu}{r^4}\right) + \frac{\mu c^2}{r^2} = 0,$$

thus

$$h^2 = \frac{\mu c^2 r^2}{r - 3\mu}.$$

The second derivative is then

$$\begin{aligned} V''(r) &= h^2\left(\frac{3}{r^4}-\frac{12\mu}{r^5}\right)-\frac{2\mu c^2}{r^3}, \\ &= \frac{\mu c^2 r^2}{r-3\mu}\left(\frac{3}{r^4}-\frac{12\mu}{r^5}\right)-\frac{2\mu c^2}{r^3}, \\ &= \frac{\mu c^2}{r^3(r-3\mu)}\left(3r-12\mu-2(r-3\mu)\right), \\ &= \frac{\mu c^2(r-6\mu)}{r^3(r-3\mu)}. \end{aligned}$$

Thus

cf Newton $\mu c^2/r^3$

$$\omega_r^2 = \left(\frac{r-6\mu}{r-3\mu}\right)\frac{\mu c^2}{r^3}.$$

NB $\omega_r^2 \to 0$ as $r \to 6\mu = 6GM/c^2$ as expected for the last circular orbit.

Therefore successive close approaches to the star (periastron) occur on a period of

$$P_r = \frac{2\pi}{\omega_r},$$

measured in terms of the proper time of the orbiting particle. During this time the azimuthal angle increases by

NB $\dot{\phi} = d\phi/d\tau \neq d\phi/dt$

$$\dot{\phi}P_r = \frac{2\pi}{\omega_r}\dot{\phi} = \frac{2\pi}{\omega_r}\frac{h}{r^2} \text{ radians.}$$

Therefore, subtracting 2π, the periastron precesses by an amount

$$\begin{aligned} \Delta\phi &= 2\pi\left[\frac{1}{r^2}\left(\frac{\mu c^2 r^2}{r-3\mu}\right)^{1/2}\left(\frac{r-3\mu}{r-6\mu}\right)^{1/2}\left(\frac{r^3}{\mu c^2}\right)^{1/2}-1\right], \\ &= 2\pi\left[\left(\frac{r}{r-6\mu}\right)^{1/2}-1\right] \text{ rads/orbit} \end{aligned}$$

If $r \gg \mu$ this can be approximated as $\delta\phi \approx 6\pi\mu/r$ rads/orbit, or

$$\boxed{\delta\phi \approx \frac{6\pi GM}{c^2 r} \text{ rads/orbit.}}$$

The precession is in the direction of the orbit (prograde).

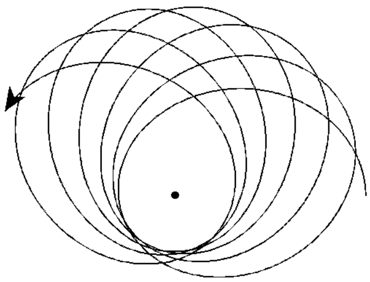

Figure: Prograde precession of an orbit started at $r = 100GM/c^2$ at its most distant point.

17.1 Precession of the perihelion of Mercury

The orbit of Mercury is observed to precess at about 5600 arcseconds/century. All but 42.98 ± 0.04 "/century can be explained by Newtonian effects – precession of the Earth's axis causing the reference frame to change (5025") and perturbations from other planets (532"). Discrepancy known in 19$^{\text{th}}$ century and ascribed to a new planet "Vulcan".

1 arcsec = 1/3600 of a degree

This bears certain similarities to "dark matter".

What does GR predict? $r_M = 5.55 \times 10^7$ km, and since $GM/c^2 = 1.47$ km

$$\Delta\phi = \frac{6\pi \times 1.47}{5.55 \times 10^7} = 0.103 \text{ arcsec/orbit}.$$

Mercury's orbital period $P_M = 0.24$ yr, so GR predicts a precession of $100 \times 0.103/0.24 = 43$ "/century!

This is one of the classic experimental tests of GR. The same effect is seen with dramatic effect in the orbits of binary pulsars where precession rates as high as 17°/year have been measured. Then used to measure the masses.

When Einstein developed GR, the anomalous precession of Mercury's orbit was the only *experimental evidence against Newton's theory. Einstein included the GR prediction in his 1916 paper presenting GR. Solving this problem so beautifully must have been supremely satisfying. Consider the beauty of GR here compared to alternatives such as altering Newton's Law of Gravity to* $1/r^{2.00000016}$ *as was also proposed ... there is no contest!*

17.2 Equations of motion for photons

The equations of motion for photons read:

$$\begin{aligned}\left(1-\frac{2\mu}{r}\right)\dot{t} &= k,\\ r^2\dot{\phi} &= h,\\ c^2\left(1-\frac{2\mu}{r}\right)\dot{t}^2-\left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2-r^2\dot{\phi}^2 &= 0.\end{aligned}$$

The only difference is the last equation which ends in c^2 for massive particles. (Remember it comes from $g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0$ for null paths.)

Substituting for $\dot{t}$ and $\dot{\phi}$ in the second equation gives an "energy" equation for photons:

$$\boxed{\dot{r}^2+\frac{h^2}{r^2}\left(1-\frac{2\mu}{r}\right)=c^2k^2.}$$

The effective potential for light is thus

$$V(r)=\frac{h^2}{2r^2}\left(1-\frac{2\mu}{r}\right).$$

The Newtonian potential term $-GM/r$ does not appear at all!

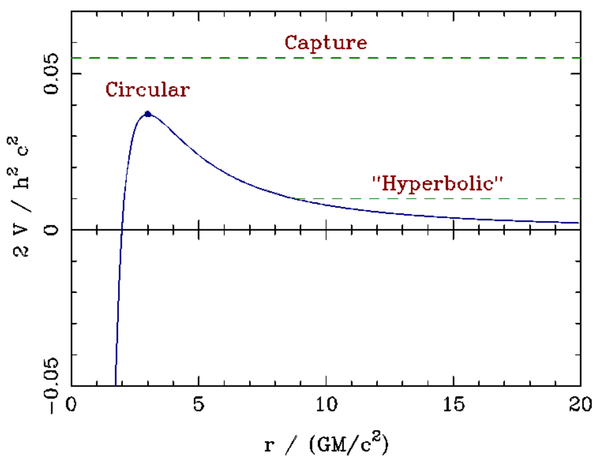

Figure: Effective potential for photons

Key points:

- Photons have equivalents of hyperbolic, circular and capture orbits.

- There are no elliptical orbits for photons.
- The circular orbits are always unstable (maximum of $V(r)$).

Lecture 18

Deflection of light

Objectives:

- *Deflection of light*

Reading: *Schutz, 10 & 11; Hobson 9 & 10; Rindler 11.*

18.1 Circular photon orbits

Show gravitational lensing pictures.

$V = h^2/2r^2(1 - 2\mu/r)$.

Circular orbits: $r = r_C$ such that $V'(r_C) = 0$, or

$$-\frac{1}{r_C^3} + \frac{3\mu}{r_C^4} = 0,$$

i.e.

$$r_C = \frac{3GM}{c^2}.$$

$3\times$ the Newtonian result $r_C = GM/c^2$, problem sheet 1.

18.2 Deflection of light by the Sun

Orbits with $r \gg GM/c^2$ suffer a small deflection which is experimentally measurable.

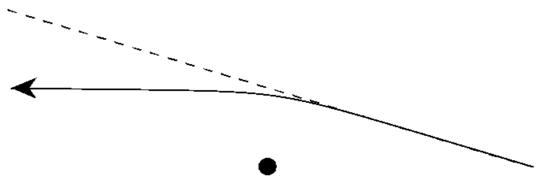

Figure: Deflection of light by a mass. Black circle shows the event horizon, so the deflection in this case is large.

To calculate light deflection, we need an equation relating r and ϕ without the affine parameter, λ. Can obtain this by noting:

$$\dot{r} = \frac{dr}{d\lambda} = \frac{dr}{d\phi}\frac{d\phi}{d\lambda} = \dot{\phi}\frac{dr}{d\phi} = \frac{h}{r^2}\frac{dr}{d\phi}.$$

Then the energy equation becomes

$$\frac{h^2}{r^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{h^2}{r^2}\left(1 - \frac{2\mu}{r}\right) = c^2k^2.$$

Making the substitution $r = 1/u$ (also used for Newtonian orbits):

$$u^4\left(-\frac{1}{u^2}\frac{du}{d\phi}\right)^2 + u^2(1 - 2\mu u) = \frac{c^2k^2}{h^2},$$

and so

$$\left(\frac{du}{d\phi}\right)^2 + u^2 - 2\mu u^3 = \frac{c^2k^2}{h^2}.$$

Finally, differentiating with respect to ϕ and dividing by $2du/d\phi$:

$$\frac{d^2u}{d\phi^2} + u = 3\mu u^2.$$

For large radii, $r \gg \mu$, $u \ll \mu^{-1}$, the RHS can be neglected and we have the SHM equation, thus:

$$u = a\sin\phi + b\cos\phi,$$

where a and b are constants, or, without loss of generality, simply

$$u = a\sin\phi,$$

or $r\sin\phi = 1/a = r_0$, a constant. This is the equation of a straight line with impact parameter r_0. As $r \to \infty$, $u \to 0$ gives $\phi = 0$ or π.

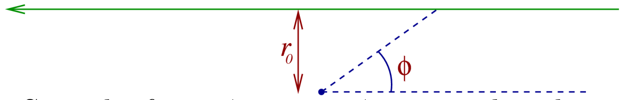

Figure: Straight from $\phi = 0$ to $\phi = \pi$ with polar equation $r \sin\phi = a^{-1}$.

Now look for a better approximation $u = u_0 + u'$ where $|u'| \ll u_0 = a \sin\phi$. Then

$$\frac{d^2u'}{d\phi^2} + u' = 3\mu u^2 \approx 3\mu u_0^2 = 3\mu a^2 \sin^2\phi = \frac{3\mu a^2}{2}(1 - \cos 2\phi),$$

neglecting small terms on the right. Particular integral is

$$u' = \frac{3\mu a^2}{2}\left(1 + \frac{1}{3}\cos 2\phi\right),$$

so a better solution is

$$u = a \sin\phi + \frac{3\mu a^2}{2}\left(1 + \frac{1}{3}\cos 2\phi\right).$$

Now $r \to \infty \implies u = 0 \implies$

$$\sin\phi = -\frac{3\mu a}{2}\left(1 + \frac{1}{3}\cos 2\phi\right) \approx -2\mu a,$$

since $\cos 2\phi \approx 1$ for $\phi = 0, \pi$. Therefore

$$\phi \approx -2\mu a, \quad \text{or} \quad \pi + 2\mu a.$$

Thus light is deflected by

$$\boxed{\Delta\phi = 4\mu a = \frac{4GM}{c^2 r_0}.}$$

This is 2× the Newtonian result (pure SR predicts zero).

For light grazing the Sun

$$\Delta\phi = \frac{4GM_\odot}{c^2 R_\odot} = \frac{4 \times 6.67 \times 10^{-11} \times 2 \times 10^{30}}{(3 \times 10^8)^2 \times 7 \times 10^8} = 8.47\times10^{-6}\,\text{rads} = 1.75\,\text{arcsec}.$$

Confirmed from observations of radio sources to 2 parts in 10^4 Deflection of light now an important tool in astronomy, "gravitational lensing". See Shapiro et al in reading.

Famously tested by British astrophysicist Eddington in 1919 using observations of stars near the Sun during a total eclipse. Made Einstein famous. Eddington the source of the well-known quote "Interviewer: Professor Eddington, is it true that only three people understand Einstein's theory? Eddington: Who is the third?"

Lecture 19

Schwarzschild Black holes

Objectives:

- *Beyond the Schwarzschild horizon*

Reading: *Schutz 11; Hobson 11; Rindler 12*

19.1 The Schwarzschild horizon

The Schwarzschild metric

$$ds^2 = c^2 \left(1 - \frac{2\mu}{r}\right) dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2 - r^2 \, d\Omega^2,$$

($d\Omega^2$ short-hand for angular terms) is singular at

$$r = R_S = 2\mu = \frac{2GM}{c^2}.$$

This is a <u>coordinate</u> singularity, similar to the singularity of the 2-sphere metric

$$ds^2 = \frac{dr^2}{1 - r^2/R^2} + r^2 \, d\theta^2,$$

when $r = R$ at the equator.

Consider radially moving particles for which $d\theta = d\phi = 0$. Then we have

$$ds^2 = g_{tt} \, dt^2 + g_{rr} \, dr^2.$$

For $r < 2\mu$, $g_{tt} < 0$, $g_{rr} > 0$. For massive particles a time-like interval $ds^2 > 0$ therefore <u>requires</u> $dr \neq 0$ and so $\dot{r}$ can never change sign.

A particle which enters the event horizon can never escape. r is time-like, t is space-like. Oblivion at $r = 0$ is the future.

Now consider photons ($ds = 0$):

$$c\,dt = \pm \left(1 - \frac{2\mu}{r}\right)^{-1} dr,$$

$+$ for outgoing, $-$ for incoming. Integrating

$$ct = \pm \int \frac{dr}{1 - 2\mu/r} = \pm \int \frac{r\,dr}{r - 2\mu} = \pm \int \left(\frac{r - 2\mu}{r - 2\mu} + \frac{2\mu}{r - 2\mu}\right) dr,$$

thus

$$ct = \pm\left(r + 2\mu \ln |r - 2\mu|\right) + \text{constant}.$$

See reading on web pages on the "Shapiro delay" for an experimental measurement of this.

Spacetime diagram:

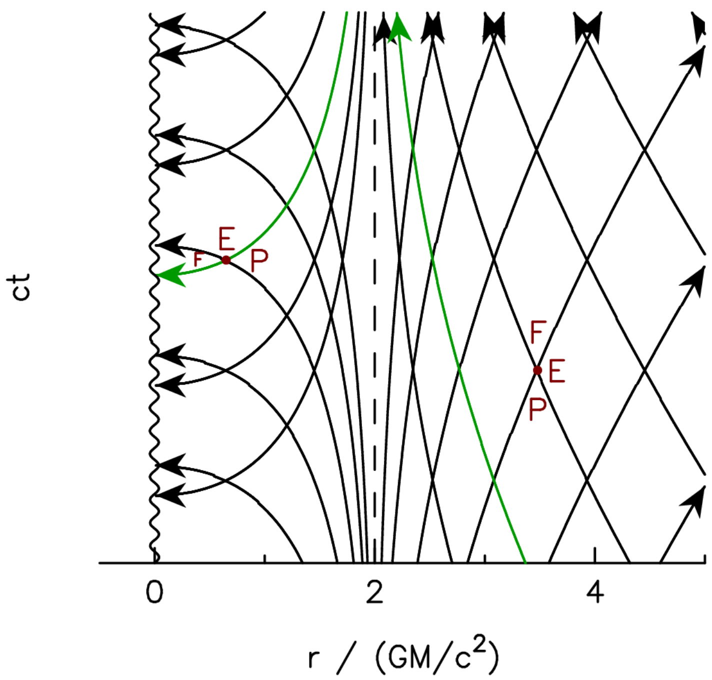

Figure: Spacetime diagram in r and t coordinates representing a series of in- and out-going photon worldlines. On the left, ingoing worldlines move *down* the ct axis. Wavy line represents the singularity at $r = 0$. The dashed line is the event horizon at $r = R_S$. The green line shows the path of the same ingoing photon on each side of $r = R_S$.

- At any event E, the future lies between the worldlines of ingoing and outgoing photons, on the same side as their direction of travel.

- As $r \to R_S$, lightcones are squeezed; worldlines take infinite coordinate time t to reach R_S.
- For $r < R_S$, lightcones are rotated and point towards $r = 0$. Particles crossing $r = R_S$ can never again be seen from $r > R_S$, thus the "event horizon".

19.2 Free-fall time

The proper time to $r = R_S$ and even to $r = 0$ is finite: for $r < R_S = 2\mu$ can write

$$c^2\,d\tau^2 = \left(\frac{2\mu}{r} - 1\right)^{-1} dr^2 - c^2\left(\frac{2\mu}{r} - 1\right) dt^2 - r^2\,d\Omega^2.$$

$d\tau$ maximum if $dt = d\Omega = 0$. Thus the maximum time one has before reaching the singularity from $r = R_S$ is

$$\tau_m = \frac{1}{c}\int_0^{2\mu}\left(\frac{2\mu}{r} - 1\right)^{-1/2} dr = \frac{\pi\mu}{c} = \frac{\pi GM}{c^3} = 15 \times 10^{-6}\left(\frac{M}{\mathrm{M}_\odot}\right)\,\mathrm{sec}\,.$$

e.g. 4.2 hours for $M = 10^9\,\mathrm{M}_\odot$. Any use of a rocket shortens this!

19.3 Kruskal-Szekeres coordinates

Schwarzschild coordinates are singular at $r = R_S$ and poor for $r < R_S$. In 1961 Kruskal found coordinates regular for all $r > 0$. Consider the incoming/outgoing photon worldlines:

$$\begin{aligned} ct &= -r - 2\mu \ln|r - 2\mu| + p, \\ ct &= +r + 2\mu \ln|r - 2\mu| + q, \end{aligned}$$

where p and q are integration constants. The idea is to use p and q to label events, i.e. as coordinates. Photon paths form a rectangular grid in (p, q) and the interval becomes

$$ds^2 = \left(1 - \frac{2\mu}{r}\right) dp\,dq - r^2\,d\Omega^2.$$

The following transform removes the awkward $1 - 2\mu/r$:

$$\begin{aligned} \bar{p} &= +\exp(p/4\mu), \\ \bar{q} &= -\exp(-q/4\mu). \end{aligned}$$

A rotation gives time- and space-like rather than null coords:

$$\begin{aligned} v &= (\bar{p} + \bar{q})/2, \\ u &= (\bar{p} - \bar{q})/2. \end{aligned}$$

These are Kruskal-Szekeres coordinates. The interval becomes

$$\boxed{ds^2 = \frac{32\mu^3}{r} e^{-r/2\mu} \left(dv^2 - du^2\right) - r^2 d\Omega^2,}$$

where

$$u^2 - v^2 = \left(\frac{r}{2\mu} - 1\right) e^{r/2\mu}.$$

Null radial paths, $ds^2 = d\Omega^2 = 0 \implies$

$$v = \pm u + \text{constant},$$

i.e. $\pm 45°$ like Minkowski!

$r = 0 \implies v^2 - u^2 = 1$, i.e. hyperbolae.

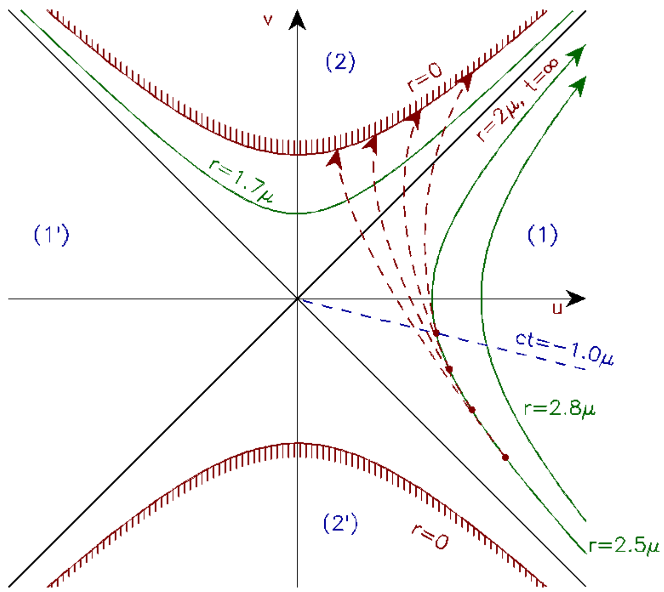

Figure: Spacetime diagram in u, v Kruskal coordinates. Light-cones now have same structure as Minkowski, so the future of any even is the region over it within 45° of the vertical.

Kruskal spacetime diagram:

- Region (1) is the region $r > R_S$ in which we live; region (2) represents $r < R_S$.
- Future of any event is contained in $\pm 45°$ "lightcone" directed upwards. Once inside region (2), the future ends on the upper $r = 0$ singularity. Can pass from (1) to (2) but not back again.
- Region (1') similar to (1) but disconnected from it: a different Universe

- Lower shaded line is a "past singularity", out of which particles emerge. Once you have entered region (2) you can never leave; once you have left (2') you can never return: a "white hole"

Whether regions 1' and 2' have any reality is unclear.

Lecture 20

The FRW metric

Objectives:

- *Friedmann-Robertson-Walker metric*

Reading: *Schutz 12; Hobson 14; Rindler 16*

20.1 Isotropy and homogeneity

On large scales, the Universe looks similar in all directions, and, in addition, assuming that ours is not a special location ("Copernican principle"), we assert that on large scales the Universe is

- isotropic: no preferred direction
- homogeneous: the same everywhere.

20.2 Cosmic time

Homogeneity implies a synchronous time t can be defined so that at a given t, physical parameters such as density and temperature are the same everywhere. Thus we can write the interval

$$ds^2 = c^2\,dt^2 - dl^2,$$

where

$$dl^2 = g_{ij}\,dx^i\,dx^j,$$

i.e. spatial terms only. $g_{0i} = 0$ because isotropy $\Longrightarrow$ no preferred direction (cf Schwarzschild). For dl^2 we look for a 3D-space of constant curvature, analagous to the surface of a sphere.

Consider the surface of a sphere in Euclidean 4D. Using Cartesian coordinates (x, y, z, w), but replacing (x, y, z) by spherical polars (ρ, θ, ϕ), we have

Using ρ for the radial coord, to save r for later; see below.

$$dl^2 = d\rho^2 + \rho^2\, d\Omega^2 + dw^2,$$

where $d\Omega^2$ is short-hand for the angular terms. Also

$$x^2 + y^2 + z^2 + w^2 = \rho^2 + w^2 = R^2,$$

and so

$$\rho\, d\rho + w\, dw = 0.$$

Therefore

$$dw^2 = \frac{\rho^2\, d\rho^2}{w^2} = \frac{\rho^2\, d\rho^2}{R^2 - \rho^2},$$

and so

$$dl^2 = d\rho^2 + \frac{\rho^2\, d\rho^2}{R^2 - \rho^2} + \rho^2\, d\Omega^2,$$

giving

$$dl^2 = \frac{d\rho^2}{1 - (\rho/R)^2} + \rho^2\, d\Omega^2.$$

This is a homogeneous, isotropic 3D space of (positive) curvature $1/R^2$. Negative and zero curvature are also possible, and setting $\rho = Rr$, all three cases can be expressed as

$$dl^2 = R^2 \left(\frac{dr^2}{1 - kr^2} + r^2\, d\Omega^2 \right),$$

where $k = -1$, 0 or +1.

In general we must allow for R to be an arbitrary function of time $R(t)$ (not position since that would destroy homogeneity), thus we arrive at

$$\boxed{ds^2 = c^2\, dt^2 - R^2(t) \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \right).}$$

This is the Friedmann-Robertson-Walker metric. It was first derived by Friedmann in 1922, and then more generally by Robertson and Walker in 1935. It applies to any metric theory of gravity, not just GR.

20.3 Geometry of the Universe

Three cases:

$k = 1$ Positive curvature, closed universe.

$k = 0$ Zero curvature, flat universe (flat space, not flat spacetime)

$k = -1$ Negative curvature, open Universe.

An alternative form of the metric is often useful. For $k = 1$, setting $r = \sin\chi$, the interval becomes

$$ds^2 = c^2\,dt^2 - R^2(t)(d\chi^2 + \sin^2\chi\,d\Omega^2).$$

The circumference of a circle of proper radius $r = \int R\,d\chi = R\chi$ is then clearly

$$C = 2\pi R\sin\chi = 2\pi R\sin\left(\frac{r}{R}\right),$$

while the area of a sphere of the same radius is

$$A = 4\pi\left(R\sin\chi\right)^2 = 4\pi R^2\sin^2\left(\frac{r}{R}\right),$$

and its volume is

$$V = \int_0^\chi \left(4\pi R^2\sin^2\chi\right) R\,d\chi = 2\pi R^3\left[\frac{r}{R} - \frac{1}{2}\sin\left(\frac{2r}{R}\right)\right].$$

As $r \to \pi R$, C and $A \to 0$, and $V \to 2\pi^2 R^3$ is finite, hence a "closed" universe, directly analogous to the surface of a sphere.

In a closed Universe you could keep travelling in one direction and yet return to where you started.

In general we can write the alternative FRW metric as

$$ds^2 = c^2\,dt^2 - R^2(t)\left(d\chi^2 + S_k^2(\chi)\,d\Omega^2\right),$$

where

$$S_k(\chi) = \begin{cases} \sin\chi, & \text{for } k = 1, \\ \chi, & \text{for } k = 0, \\ \sinh\chi, & \text{for } k = -1. \end{cases}$$

20.4 Redshift

The wavelength of light from astronomical sources is a crucial, easily measured observable. Consider two pulses of light emitted at times $t = t_e$ and $t = t_e + \delta t_e$ by an object at χ towards an observer at the origin who picks them up at $t = t_o$ $t = t_o + \delta t_o$.

For photons travelling towards the origin, since $ds = 0$

$$c\,dt = -R(t)\,d\chi,$$

Therefore

$$\chi = \int_{t_e}^{t_o} \frac{c\,dt}{R(t)} = \int_{t_e+\delta t_e}^{t_o+\delta t_o} \frac{c\,dt}{R(t)}.$$

Subtracting the first integral from the second:

$$\int_{t_o}^{t_o+\delta t_o} \frac{c\,dt}{R(t)} - \int_{t_e}^{t_e+\delta t_e} \frac{c\,dt}{R(t)} = 0.$$

For small intervals $R(t)$ is almost constant, so

$$\frac{\delta t_o}{R(t_o)} = \frac{\delta t_e}{R(t_e)}.$$

Therefore the redshift z is given by

$$1 + z = \frac{\lambda_o}{\lambda_e} = \frac{\nu_e}{\nu_o} = \frac{\delta t_o}{\delta t_e} = \frac{R(t_o)}{R(t_e)}.$$

$1 + z$ is thus the factor by which the Universe has expanded in between emission and reception of the light.

20.5 Hubble's Law

The universal "fluid" (= galaxies) is at rest in comoving coordinates r or χ, θ and ϕ. Expansion of the Universe is contained in the size factor $R(t)$.

Consider the proper distance to a galaxy at radius χ

$$d_P = \int_0^{\chi} R(t)\,d\chi = R(t)\chi,$$

Since χ is fixed, the rate of recession of the galaxy is

$$v = \frac{d}{dt}(d_P) = \dot{R}\chi = \frac{\dot{R}}{R} d_P.$$

Identifying

$$H(t) = \dot{R}/R,$$

we have

$$\boxed{v = H(t) d_P}$$

which is Hubble's Law, while $H(t)$ is Hubble's "constant" $= H(t_0) = H_0$ today.

Hubble's Law is thus a direct outcome of homogeneity and isotropy.

Lecture 21

Dynamics of the Universe

Objectives:

- *The Friedmann equations*

Reading: *Schutz 12; Hobson 14; Rindler 16*

21.1 Friedmann's equation

The evolution of the Universe in GR is determined as follows:

1. The FRW interval $\Longrightarrow$ the metric, e.g. $g_{rr} = -R^2/(1 - kr^2)$
2. The metric $\Longrightarrow$ $\Gamma^{\alpha}{}_{\beta\gamma}$, the connection.
3. The metric and connection $\Longrightarrow$ $R_{\alpha\beta}$, the Ricci tensor.
4. The Ricci tensor and field equations $\Longrightarrow$ differential equations for the size factor R and the fluid density ρ.

Jumping straight in at step 4, consider See handout 7

$$R_{tt} = 3\frac{\ddot{R}}{R}.$$

Use field equations in the form

$$R_{\alpha\beta} = k\left(T_{\alpha\beta} - \frac{1}{2}T g_{\alpha\beta}\right).$$

Assume perfect fluid:

$$T_{\alpha\beta} = \left(\rho + \frac{p}{c^2}\right) U_\alpha U_\beta - p g_{\alpha\beta}.$$

Fluid is static in co-moving coordinates of FRW metric so $U^i = 0$ and

$$g_{\alpha\beta}U^\alpha U^\beta = g_{tt}U^t U^t = c^2,$$

so since $g_{tt} = c^2$, $U^t = 1$. Hence

$$U_t = g_{tt}U^t = c^2,$$

and

$$T_{tt} = \left(\rho + \frac{p}{c^2}\right)c^4 - pc^2 = \rho c^4,$$

while

$$T = g_{\alpha\beta}T^{\alpha\beta} = \left(\rho + \frac{p}{c^2}\right)g_{\alpha\beta}U^\alpha U^\beta - pg_{\alpha\beta}g^{\alpha\beta} = \left(\rho + \frac{p}{c^2}\right)c^2 - 4p = \rho c^2 - 3p.$$

Therefore

$$3\frac{\ddot{R}}{R} = k\left(\rho c^4 - \frac{1}{2}(\rho c^2 - 3p)c^2\right).$$

Putting $k = -8\pi G/c^4$ we obtain

$$\boxed{\ddot{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)R.} \qquad (21.1)$$

This is the acceleration equation.

Similarly the rr, $\theta\theta$ and $\phi\phi$ components all lead to:

$$\boxed{\dot{R}^2 + kc^2 = \frac{8\pi G}{3}\rho R^2.} \qquad (21.2)$$

This is the Friedmann equation.

Finally, taking the time derivative of the Friedmann equation and substituting for $\ddot{R}$ from the acceleration equation it is simple to show: Prove this

$$\boxed{\dot{\rho} + \frac{3\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right) = 0.} \qquad (21.3)$$

which is the fluid equation. Alternatively this comes from $T^{\alpha\beta}{}_{;\alpha} = 0$.

21.1.1 Newtonian interpretation

Each of Eqs 21.1, 21.2 and 21.3 has an approximate Newtonian interpretation. If one considers an expanding uniform density sphere then

$$\ddot{R} = -\frac{4\pi G}{3}\rho R.$$

There is no Newtonian explanation for the pressure term in the acceleration equation. Conserving energy for a particle on the edge of such a sphere gives:

$$\frac{1}{2}\dot{R}^2 - \frac{4\pi G}{3}\rho R^2 = \frac{E}{m}.$$

Newtonian equivalent for curvature term kc^2 is total energy per unit mass. Finally the fluid equation follows directly from

$$T\,dS = dU + p\,dV,$$

setting $dS = 0$ (reversible adiabatic, no temperature gradients) and using mass–energy equivalence. Such Newtonian interpretations are a fudge: Eqs 21.1, 21.2 and 21.3 are relativistic.

21.2 The cosmological constant

In 1917 Einstein modified the field equations to read

$$R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta} + \Lambda g^{\alpha\beta} = kT^{\alpha\beta},$$

where Λ is the cosmological constant. Still satisifes $T^{\alpha\beta}{}_{;\alpha} = 0$ since $g^{\alpha\beta}{}_{;\gamma} = 0$. Nowadays, it is usual to place the new term on the right as the stress–energy tensor of the vacuum.

$$R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta} = k\left(T^{\alpha\beta} - \frac{\Lambda}{k}g^{\alpha\beta}\right).$$

Second term in brackets on the right has the form of a perfect fluid

$$\left(\rho_\Lambda + \frac{p_\Lambda}{c^2}\right)U^\alpha U^\beta - p_\Lambda g^{\alpha\beta},$$

if

$$\rho_\Lambda + \frac{p_\Lambda}{c^2} = 0,$$

and

$$p_\Lambda = \frac{\Lambda}{k} = -\frac{\Lambda c^4}{8\pi G},$$

and thus

$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi G}.$$

i.e. a fluid of constant density and negative pressure.

This is "dark energy", perhaps the most puzzling problem in modern physics.

21.2.1 Einstein's static universe

Negative pressure allows a static Universe. From

$$\ddot{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)R,$$

$\ddot{R}$ can be zero if

$$\rho + \frac{3p}{c^2} = 0.$$

Here ρ and p are the sums of contributions from all components. Considering matter and Λ only, for matter $p_M \ll \rho_M c^2$ so

$$\rho + \frac{3p}{c^2} \approx \rho_M + \rho_\Lambda + \frac{3p_\Lambda}{c^2} = \rho_M - 2\rho_\Lambda.$$

Thus

$$\ddot{R} = -\frac{4\pi G}{3}\left(\rho_M - 2\rho_\Lambda\right)R,$$

which is zero if $\rho_M = 2\rho_\Lambda$. This is Einstein's static universe. Unfortunately it would not be static for long since it is unstable. Consider a perturbation $\rho_M = 2\rho_\Lambda + \rho'$, $R = R_0 + R'$. To first order

$$\ddot{R}' = -\frac{4\pi G}{3}\rho' R_0.$$

If $R' > 0$ we expect $\rho' < 0$ since matter is diluted as the universe expands, hence $\ddot{R}' > 0$ and the perturbation will grow $\implies$ instability. The universe either contracts or expands away from $R = R_0$.

Λ therefore can give a static but not a stable universe. Had Einstein realised this, he could have predicted an expanding or contracting universe. Perhaps this was why he once referred to the cosmological constant as "my greatest blunder" (as quoted by Gamow, 1970).

Lecture 22

Cosmological distances

Objectives:

- *Friedmann-Robertson-Walker metric*

Reading: *Schutz 12; Hobson 14 and 15; Rindler 17*

22.1 Distances

There is no one "distance" in cosmology. Using the metric

$$ds^2 = c^2\, dt^2 - R^2(t) \left(d\chi^2 + S^2(\chi)\, d\Omega^2\right),$$

the easiest to define is the ruler or proper distance d_P

$$\boxed{d_P = R_0 \chi,}$$

where R_0 is the present size factor of the Universe.

A more practical measure is the luminosity distance defined as the distance at which the observed flux f from an object equals the standard Euclidean formula:

$$f = \frac{L}{4\pi d_L^2},$$

where L is the luminosity.

Consider a source S at the origin (can always shift origin) and an observer O at χ. When light reaches O at time t_o, it is spread equally (isotropy) over an area

$$A = 4\pi R_0^2 S^2(\chi).$$

The flux observed is therefore

$$f = \frac{L}{4\pi R_0^2 S^2(\chi)(1+z)^2}.$$

The $(1+z)^2$ factor comes from the redshift which reduces both the energy and arrival rate of the photons. The $R^2(t)S^2(\chi)$ comes from the angular terms of the FRW metric. Therefore

$$\boxed{d_L = R_0 S_k(\chi)(1+z).}$$

The angular diameter distance d_A is defined such that

$$\alpha = \frac{l}{d_A},$$

where α is the angle subtended by an object of size l.

Sketch:

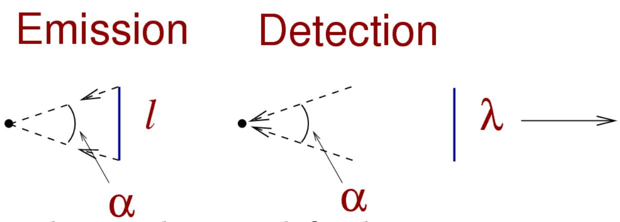

Figure: The angular size defined at emission is preserved during expansion because the photons travel along radial paths towards the origin.

Photons travel from source to observer along radial paths. Angular size defined at time of emission. From the FRW metric,

$$l = R(t_e)S_k(\chi)\alpha,$$

and therefore

$$\boxed{d_A = R(t_e)S_k(\chi) = \frac{R_0 S_k(\chi)}{1+z},}$$

since

$$1 + z = \frac{R_0}{R(t_e)}.$$

In each case we need χ which is connected to the time of emission t_e and observation t_0 through

$$\chi = \int_{t_e}^{t_0} \frac{c\,dt}{R(t)}.$$

We can replace t by z where

$$1 + z = \frac{R_0}{R},$$

so

$$dz = -\frac{R_0}{R^2}\dot{R}\,dt,$$

and hence

$$\chi = -\int_z^0 \frac{cR^2}{R_0\dot{R}}\frac{1}{R}\,dz,$$

so, remembering $H = \dot{R}/R$,

$$R_0\chi = \int_0^z \frac{c\,dz}{H(z)}.$$

Thus χ, and hence the distances, are sensitive to the expansion history of the Universe encoded in $H(z)$. e.g. flux vs redshift ("Hubble diagrams") of supernovae $\Longrightarrow$ a cosmological constant.

22.2 The future of our Universe

We now believe that our Universe is 74% cosmological constant, 26% matter (5% baryonic). In the future Λ will dominate since $\rho_M \propto R^{-3}$ while ρ_Λ is constant, so the Friedman equation tends to

$$\dot{R} = \left(\frac{8\pi G}{3}\rho_\Lambda\right)^{1/2} R,$$

the curvature term being constant becomes negligible compared to the above terms. This equation describes a de Sitter universe in which there is only a cosmological constant. Clearly

$$R = R_0 \exp(t/\tau),$$

where t is measured from the present and

$$\tau = \left(\frac{3}{8\pi G\rho_\Lambda}\right)^{1/2} = 1.6 \times 10^{10}\,\mathrm{yr},$$

for our Universe.

Outrunning a photon: consider a photon emitted at time $t = t_e$ (counting from the present). By time t it will have reached comoving radius χ given by

$$\chi(t) = \int_{t_e}^t \frac{c\,dt}{R(t)} = \frac{c}{R_0}\int_{t_e}^t e^{-t/\tau}\,dt = \frac{c\tau}{R_0}\left(e^{-t_e/\tau} - e^{-t/\tau}\right).$$

As $t \to \infty$, the photon has reached a proper distance as measured in today's Universe ($R = R_0$) of

$$d_P = R_0 \chi = c\tau e^{-t_e/\tau}.$$

<u>Implication:</u> photons in a de Sitter unverse never catch up distant parts of the Universe. The later a photon is emitted, the shorter the distance it travels in today's terms. Put differently, we see no photons that a galaxy at proper distance d_P emitts after a time

$$t_e = \tau \ln \frac{c\tau}{d_P}.$$

Were we to observe a clock in such a galaxy, we would see it get slower and slower, never quite making it to t_e. The galaxy meanwhile becomes increasing redshifted and ever fainter. This is an external event horizon in fact.

As a consequence, in the future, all galaxies now in the Hubble flow away from us will disappear from our view, unless the "dark energy" driving the expansion runs out of steam.

Lecture 23

Linear GR

Objectives:

- *Linearised GR*

Reading: *Schutz 8; Hobson 17; Rindler 15*

23.1 Approximating GR

The non-linearity of GR makes it difficult to solve in most situations. It is useful to develop an approximate form of the field equations for the common case of weak fields.

In weak fields we can assume that there are coordinates x^α in which the metric can be written

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta},$$

where $|h_{\alpha\beta}| \ll 1$. Using this the field equations

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = kT_{\alpha\beta},$$

can be approximated to first order in h.

e.g. the connection

$$\begin{aligned} \Gamma^\alpha{}_{\beta\gamma} &= \frac{1}{2} g^{\alpha\delta} \left(g_{\delta\gamma,\beta} + g_{\beta\delta,\gamma} - g_{\beta\gamma,\delta} \right), \\ &= \frac{1}{2} \eta^{\alpha\delta} \left(h_{\delta\gamma,\beta} + h_{\beta\delta,\gamma} - h_{\beta\gamma,\delta} \right), \end{aligned}$$

is first-order in h, so the Riemann tensor boils down to

$$R^\rho{}_{\alpha\beta\gamma} = \Gamma^\rho{}_{\alpha\gamma,\beta} - \Gamma^\rho{}_{\alpha\beta,\gamma}.$$

Eventually one finds:

Do not try to remember this!

$$h_{,\alpha\beta} + \Box h_{\alpha\beta} - \eta^{\gamma\delta}\left(h_{\alpha\gamma,\delta\beta} + h_{\delta\beta,\alpha\gamma}\right) - \left(\Box h - h^{\sigma\rho}{}_{,\sigma\rho}\right)\eta_{\alpha\beta} = 2kT_{\alpha\beta},$$

where $h = \eta^{\alpha\beta}h_{\alpha\beta}$ and

$$\Box = \eta^{\sigma\rho}\partial_\sigma\partial_\rho = \partial_\sigma\partial^\sigma = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2,$$

is the D'Alembertian or wave operator.

23.2 Lorenz Gauge

The choice of $h_{\alpha\beta}$ is not unique; it depends on the underlying coordinates. This can be used to simplify the linearised equations. For instance consider the coordinate transform

$$x'^\alpha = x^\alpha + \epsilon^\alpha,$$

with ϵ^α and its derivatives $\ll 1$ (easier here not to put primes on indices; $h^{\alpha\beta}$ is not a tensor). Then

$$\begin{aligned} g_{\alpha\beta} &= \frac{\partial x'^\gamma}{\partial x^\alpha}\frac{\partial x'^\delta}{\partial x^\beta}g'_{\gamma\delta}, \\ &= \left(\delta^\gamma_\alpha + \epsilon^\gamma{}_{,\alpha}\right)\left(\delta^\delta_\beta + \epsilon^\delta{}_{,\beta}\right)g'_{\gamma\delta}, \end{aligned}$$

so

$$\eta_{\alpha\beta} + h_{\alpha\beta} = \left(\delta^\gamma_\alpha + \epsilon^\gamma{}_{,\alpha}\right)\left(\delta^\delta_\beta + \epsilon^\delta{}_{,\beta}\right)\left(\eta_{\gamma\delta} + h'_{\gamma\delta}\right).$$

Thus

$$\eta_{\alpha\beta} + h_{\alpha\beta} = \eta_{\alpha\beta} + \epsilon^\delta{}_{,\beta}\eta_{\alpha\delta} + \epsilon^\gamma{}_{,\alpha}\eta_{\gamma\beta} + h'_{\alpha\beta},$$

and so

$$h'_{\alpha\beta} = h_{\alpha\beta} - \epsilon_{\alpha,\beta} - \epsilon_{\beta,\alpha}.$$

Very similar to gauge transformation of EM where the physics is invariant to transforms of the 4-potential of the form

$$A'_\alpha = A_\alpha + \psi_{,\alpha},$$

where ψ is some scalar field.

Choose ϵ^α to simplify field equations. In particular choosing coordinates such that

$$h^{\alpha\beta}{}_{,\beta} = \frac{1}{2}\eta^{\alpha\beta}h_{,\beta},$$

("Lorenz gauge"), then the field equations reduce to

$$\Box h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\Box h = 2kT_{\alpha\beta}.$$

Further simplification comes from defining

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2} h \eta_{\alpha\beta},$$

("trace reversal" since $\bar{h} = -h$). The Lorenz gauge becomes

$$\boxed{\bar{h}^{\alpha\beta}{}_{,\beta} = 0,}$$

while the field equations reduce to

$$\boxed{\Box \bar{h}_{\alpha\beta} = 2kT_{\alpha\beta},}$$

or in full:

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \bar{h}_{\alpha\beta} = -\frac{16\pi G}{c^4} T_{\alpha\beta}.$$

There is still some remaining freedom: the same relations survive coordinate transforms $x'^{\alpha} = x^{\alpha} + \epsilon^{\alpha}$ provided

$$\Box \epsilon^{\alpha} = 0.$$

23.3 Newtonian limit [not in lectures]

Consider a time-independent, weak-field. Setting $k = -8\pi G/c^4$, and $\Box = -\nabla^2$, the field equations become

$$\boxed{\nabla^2 \bar{h}^{\alpha\beta} = \frac{16\pi G}{c^4} T^{\alpha\beta},}$$

which has the form of Poisson's equation. If all mass is stationary, then only $T^{00} = \rho c^2$ is significant so we have

$$\nabla^2 \bar{h}^{00} = \frac{16\pi G \rho}{c^2},$$

and by analogy with

$$\nabla^2 \phi = 4\pi G \rho,$$

we can immediately write

$$\bar{h}^{00} = \frac{4\phi}{c^2},$$

where ϕ is the Newtonian potential. All other components $= 0$.

From this we deduce $h = -\bar{h} = -4\phi/c^2$, and since

$$h^{\alpha\beta} = \bar{h}^{\alpha\beta} + \frac{1}{2} h \eta^{\alpha\beta},$$

we find

$$h^{00} = h^{11} = h^{22} = h^{33} = \frac{2\phi}{c^2},$$

Finally, since $g^{\alpha\beta} = \eta^{\alpha\beta} + h^{\alpha\beta}$, and lowering indices we find

$$ds^2 = c^2 \left(1 + \frac{2\phi}{c^2}\right) dt^2 - \left(1 - \frac{2\phi}{c^2}\right) \left(dx^2 + dy^2 + dz^2\right).$$

This approximate metric is useful for studying gravitational lensing around anything more complex than a point mass, e.g. a star plus planets, or clusters of galaxies.

Lecture 24

Gravitational waves

Objectives:

- *Linearised GR*

Reading: *Schutz 9; Hobson 17; Rindler 15*

24.1 Gravitational waves

In the vacuum, $T^{\alpha\beta} = 0$, and so

$$\Box \bar{h}^{\alpha\beta} = \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \bar{h}^{\alpha\beta} = 0.$$

This is the wave equation for waves that travel at the speed of light c. It has solution

$$\bar{h}^{\alpha\beta} = A^{\alpha\beta} \exp(i k_\rho x^\rho).$$

Remembering that

$$\Box = \eta^{\rho\sigma} \partial_\rho \partial_\sigma,$$

and substituting the solution into the wave equation gives

$$\eta^{\rho\sigma} k_\rho k_\sigma \bar{h}^{\alpha\beta} = 0.$$

For non-zero solutions we must have

$$\eta^{\rho\sigma} k_\rho k_\sigma = k^\sigma k_\sigma = 0,$$

i.e. $\vec{k}$ is a null vector. This is the wave vector and usually written $\vec{k} = (\omega/c, \mathbf{k})$. $k^\sigma k_\sigma = 0$ is then just the familiar $\omega = ck$.

24.2 Gauge conditions

Our solution must satisfy the Lorenz gauge

$$\bar{h}^{\alpha\beta}{}_{,\beta} = 0,$$

which leads to the four conditions:

$$A^{\alpha\beta} k_\beta = 0. \tag{24.1}$$

Four more conditions come from our freedom to make coordinate transformations with any vector field ϵ^α satisfying

$$\Box \epsilon^\alpha = 0.$$

This allows us to remove waves in the coordinates.

The standard choice is called the transverse–traceless (TT) gauge in which

$$\eta_{\alpha\beta} A^{\alpha\beta} = 0, \tag{24.2}$$

which makes $A^{\alpha\beta}$ traceless, and

$$A^{ti} = 0. \tag{24.3}$$

Eq. 24.1 can be written as

$$A^{\alpha t} k_t + A^{\alpha i} k_i = 0,$$

and setting $\alpha = t$, Eq. 24.3 $\implies$ $A^{tt} = 0$, thus $A^{t\alpha} = A^{\alpha t} = 0$.

Specialising to a wave in the z-direction, $k_\alpha = (k_t, 0, 0, k_z)$, then Eq. 24.1 shows that

$$A^{\alpha t} k_t + A^{\alpha z} k_z = A^{\alpha z} k_z = 0,$$

so

$$A^{\alpha z} = 0,$$

hence "transverse". Finally, since $A^{tt} = A^{zz} = 0$, Eq. 24.2 shows that

$$A^{xx} + A^{yy} = 0,$$

and so

$$A^{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where a and b are arbitrary constants.

> The 2 degrees of freedom represented by a and b correspond to 2 polarisations of gravitational waves.

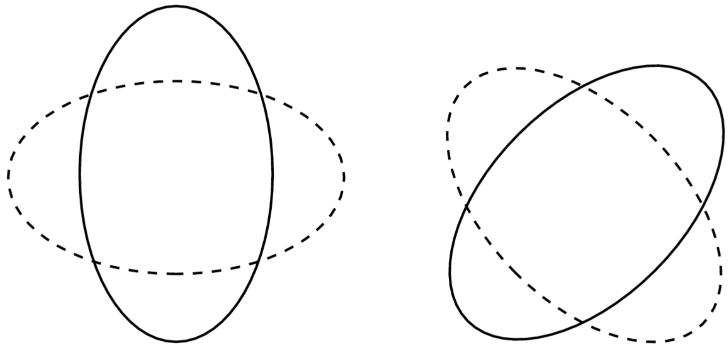

Figure: The two polarisations can be separated into tidal distortions at 45° to each other. The figure shows the extremes of the distortion that occur to a ring of freely floating particles as a gravitational wave passes (directly in or out of the page). The extent of the distortion is *very* exaggerated compared to reality!

The two polarisations give varying tidal distortions perpendicular to the direction of travel.

24.3 Generation of gravitational waves

The equation

$$\Box \bar{h}^{\alpha\beta} = 2kT^{\alpha\beta}$$

is analagous to the equation in the Lorenz gauge in EM

$$\Box \phi = \frac{\rho}{\epsilon_0},$$

which has solution

$$\phi(t, \mathbf{r}) = \int \frac{[\rho]}{4\pi\epsilon_0 R} dV,$$

where $[\rho] = \rho(t - R/c, \mathbf{x})$, $R = |\mathbf{r} - \mathbf{x}|$. Thus by analogy:

$$\bar{h}^{\alpha\beta} = 2k \int \frac{[T^{\alpha\beta}]}{4\pi R} dV$$

If the origin is inside the source, and $|\mathbf{r}| = r \gg |\mathbf{x}|$ (compact source), we are left with the far-field solution

$$\bar{h}^{\alpha\beta}(t, \mathbf{r}) \approx \frac{2k}{4\pi r} \int T^{\alpha\beta}(t - r/c, \mathbf{x}) \, dV.$$

Using the energy-momentum conservation relation ${T^{\alpha\beta}}_{,\beta} = 0$ one can then show that

Q16, problem sheet 2

$$\bar{h}^{ij} \approx -\frac{2G}{c^4 r} \frac{d^2 I^{ij}}{dt^2},$$

where

$$I^{ij} = \int \rho x^i x^j \, dV,$$

is the moment-of-inertia or quadrupole tensor.

No gravitational dipole radiation because conservation of momentum means that $\int \rho x^i \, dV$ is constant.

24.3.1 Estimate of wave amplitude

Consider two equal masses M separated by a in circular orbits in the x–y plane of angular frequency Ω around their centre of mass. Then

$$I^{xx} = \int \rho x^2 \, dV = 2M \left(\frac{a}{2} \cos \Omega t \right)^2 = \frac{1}{4} M a^2 \left(1 + \cos 2\Omega t \right).$$

Differentiating twice gives

$$\bar{h}^{xx} = \frac{2GMa^2\Omega^2}{c^4 r} \cos 2\Omega t.$$

Other terms similar. Consequences:

- Gravitational wave has twice frequency of the source (quadrupole radiation)
- Amplitude $\sim GMa^2\Omega^2/c^4 r$.

Example: $M = 10\,\mathrm{M}_\odot$, $a = 1\,\mathrm{R}_\odot$, at $r = 8\,\mathrm{kpc}$ (Galactic centre). Then Kepler3

$$\Omega^2 = \frac{G(M_1 + M_2)}{a^3} = 7.8 \times 10^{-4}\,\mathrm{rad}^2\,\mathrm{s}^{-2}.$$

(Orbital period 38 mins, GW period 19 mins).

Find $h \sim 2 \times 10^{-21}$. This is a tiny distortion of space, $< 0.1\,\mathrm{mm}$ in the distance from us to the nearest star.

Lecture 25

Detection of gravitational waves

Objectives:

- *GRW detection*

Reading: *Schutz 9; Hobson 17; Rindler 15*

25.1 Detecting Gravitational waves

The decreasing orbital period of binary pulsar provides strong but indirect evidence of gravitational waves. Direct detection of gravitational waves is one of the greatest challenges of modern experimental physics. The main possible sources are:

- Very close pairs of stars: white dwarfs, neutron stars and black-holes in orbits of a few minutes.
- Mergers of super-massive black-holes at the centres of galaxies. Most powerful events of all – $\sim 4\%$ of total mass in gravitational waves. e.g. could release $\sim 10^7\,\mathrm{M}_\odot$ of energy within about an hour, $L \sim 10^{24}\,\mathrm{L}_\odot \gg$ rest of observable Universe!
- Asymmetric rapidly rotating neutron stars, e.g. in X-ray binaries.
- Supernovae
- Fluctuations of the very early Universe

GWR can give a completely new view of these exotic targets, and could provide the first ever test of GR in the strong field $\phi \sim c^2$ regime.

25.2 Detectors

Two types:

1. Resonant bars (Joseph Weber, 1960s).
2. Michelson interferometers (suspended mirrors act as test masses). Mirrors $> 99.999\%$ reflection. Existing (main ones):
   (a) LIGO: 2 interferometers in the USA with 4 km long arms
   (b) VIRGO: France/Italy, 3 km arms
   (c) GEO600: Germany/UK, 600 m arms

   Planned: LISA, 2 million km space-based interferometer.

Multiple detectors vital for believable result.

25.3 Ground-based detection

LIGO: 4 km-long arms $\implies$ detect $\Delta l \sim 10^{-18}$ m for $h \sim 10^{-21}$.

Advantages:

- Short arms good for high-frequency inspirals. e.g. neutron star pairs reach ~ 1 kHz.
- High laser power possible.
- Can be upgraded.

Disadvantages:

- Seismic noise limits low frequencies, so most common sources undetectable
- Short arms require very high precision
- Events are very short lived (< 1 second), making them hard to detetect

Current LIGO can detect merging neutron stars out to 10 Mpc. However, no detection to date: such events are probably rare.

Advanced LIGO will raise max distance to 100 Mpc, $1000\times$ increase in volume. Expect several events per year.

25.4 Space-based detection

Space offers:

- Potentially long interferometer arms
- No seismic noise so sensitive to much lower frequencies, e.g. early Universe, merger of supermassive black-holes, early detection of lower mass mergers and commoner types of binary star.

but

- low laser power limits high frequency sensitivity.

LISA is a proposed interferometer with spacecraft 2 million km apart.

25.5 Numerical relativity

At low signal-to-noise, one needs to know the shape of the waveform to detect it. Thus computer simulations are part of the detection effort. Good progress has been made in understanding the merger of two black-holes.

Prospects for the first direct detection are good; its now down to the Universe to give us some observable events.

Watch this space!