Lecture 16: Schwartzschild Orbits

Reading: Schutz - ch 11

Hobson - ch 9

Rindler - ch 11

Lecture 15 Summary

- Schwartzschild radius: $R_s = 2GM/c^2$
  - Within this radius there are no stationary observers
  - Massive particles can only exist by moving radially inwards
  - They end up on the central singularity
  - Often $R_s < R$ and so the metric does not apply

Lecture 15 Summary

- Schwartzschild Equations of Motion

$$ds^2 = c^2\left(1 - \frac{2\mu}{r}\right)dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1}dr^2 - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

- Apply Euler-Lagrange eqns:

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) - \frac{\partial L}{\partial x^\alpha} = 0.$$

- Gives (for θ=π/2, constant):

$$\left(1 - \frac{2\mu}{r}\right)\dot{t} = k \qquad\qquad r^2\dot{\phi} = h.$$

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- Apply either of:

  – $c^2 = ds^2/d\tau^2$ or $\vec{U}\cdot\vec{U} = g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = \text{constant}$ $= c^2$

- And substitute for t and φ to get (for massive particles):

$$\boxed{\dot{r}^2 + \frac{h^2}{r^2}\left(1-\frac{2\mu}{r}\right) - \frac{2\mu c^2}{r} = c^2\left(k^2-1\right).}$$

The Schwartzschild Energy Equation

$$\boxed{\dot{r}^2 + \frac{h^2}{r^2}\left(1 - \frac{2\mu}{r}\right) - \frac{2\mu c^2}{r} = c^2\left(k^2 - 1\right).}$$

- This is an energy equation in the form:

  kinetic energy + potential energy = constant

- Compare this with the Newtonian energy equation:

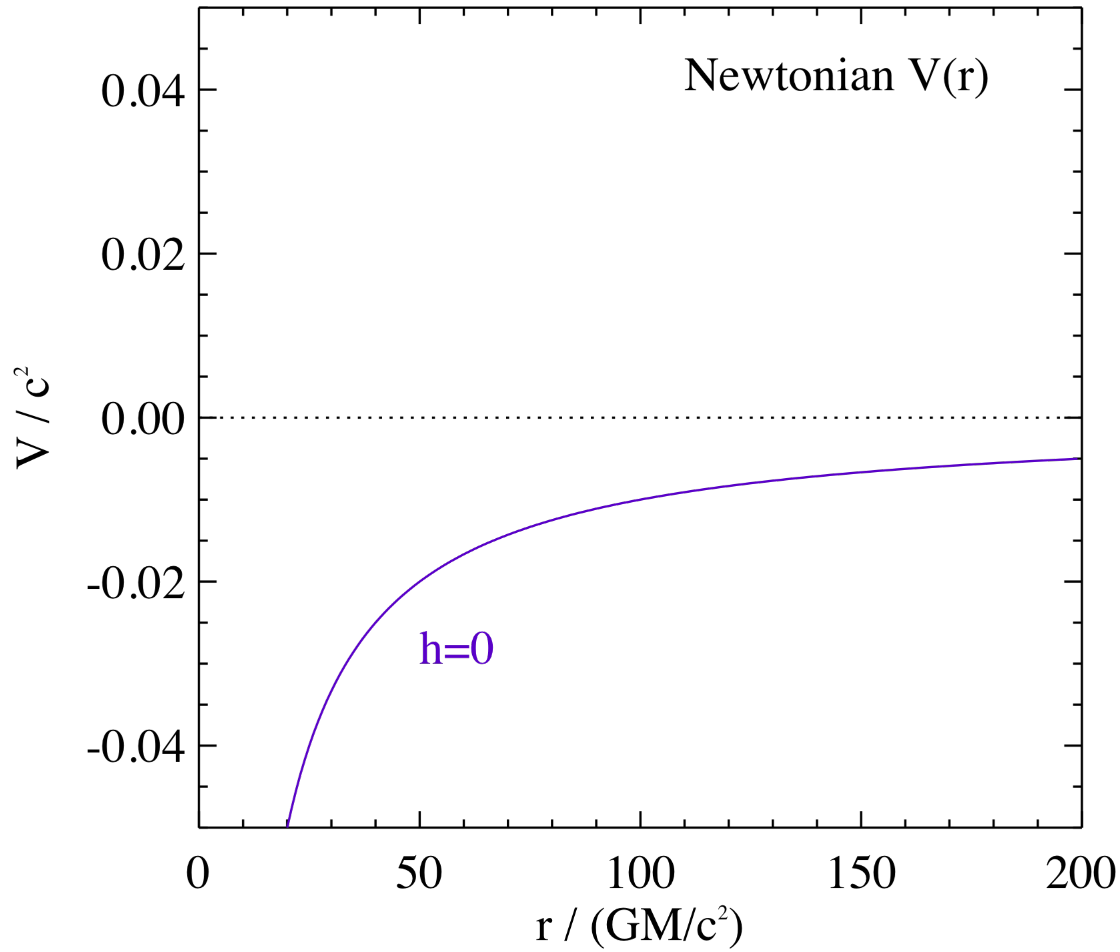
$$\dot{r}^2 + \frac{h^2}{r^2} - \frac{2GM}{r} = \frac{2E}{m}$$

GR introduces a term in r $^{-3}$ to the potential

Behaviour of the Newtonian Potential

$$V_N(r) = \frac{h^2}{2r^2} - \frac{GM}{r}$$

h in units of μc



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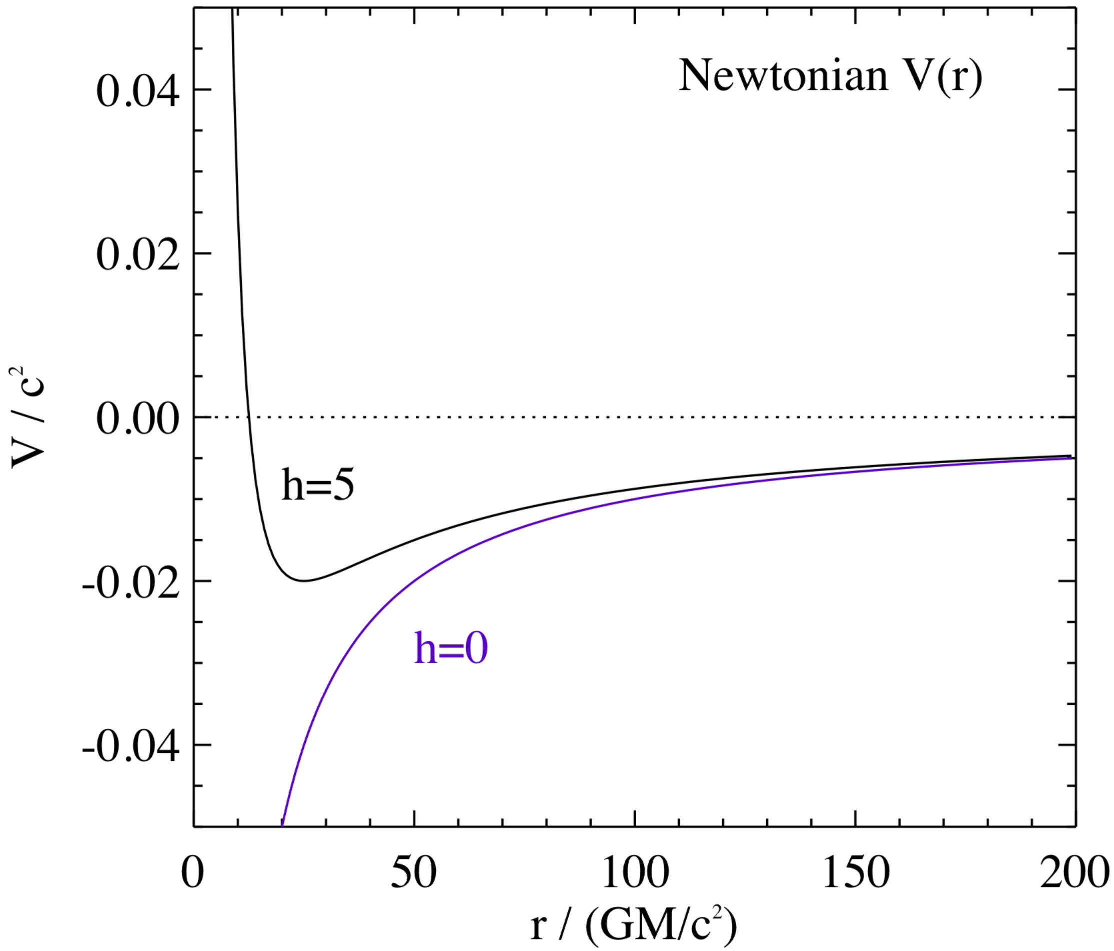

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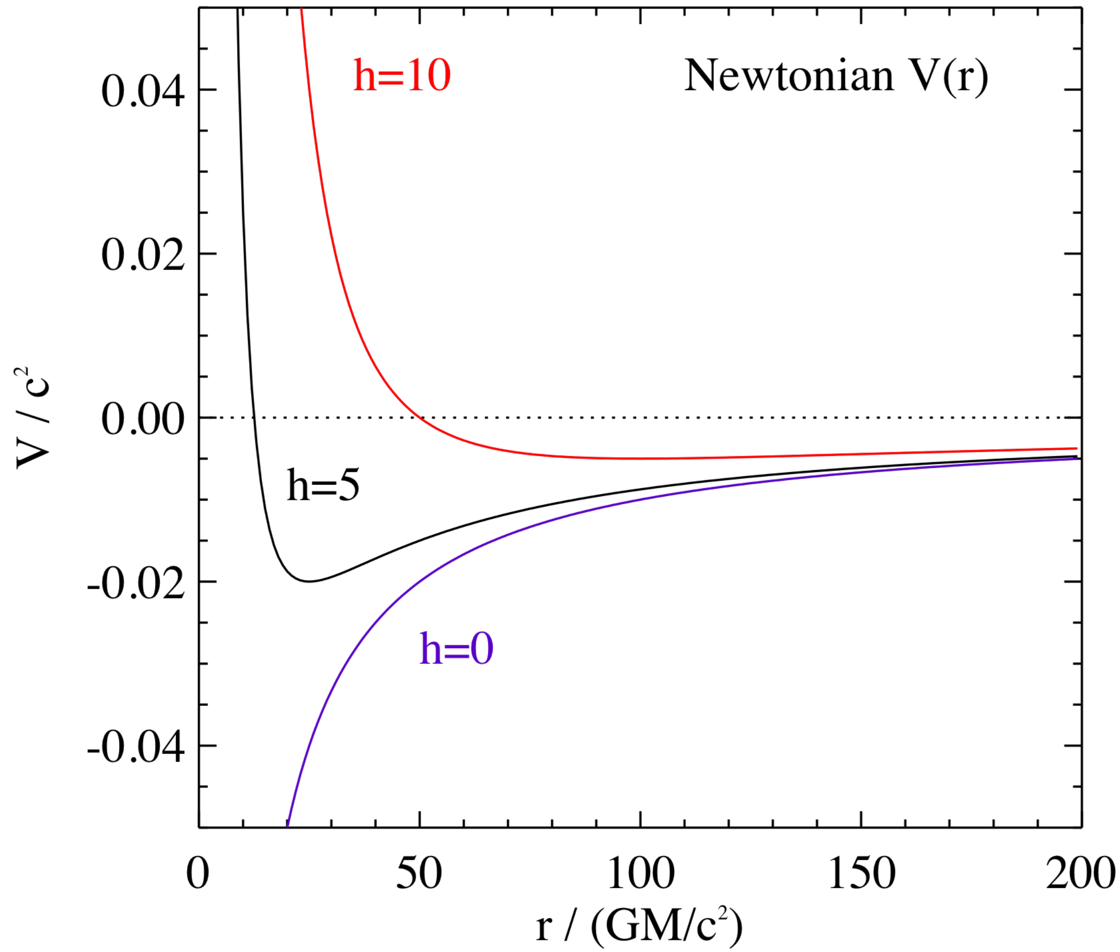

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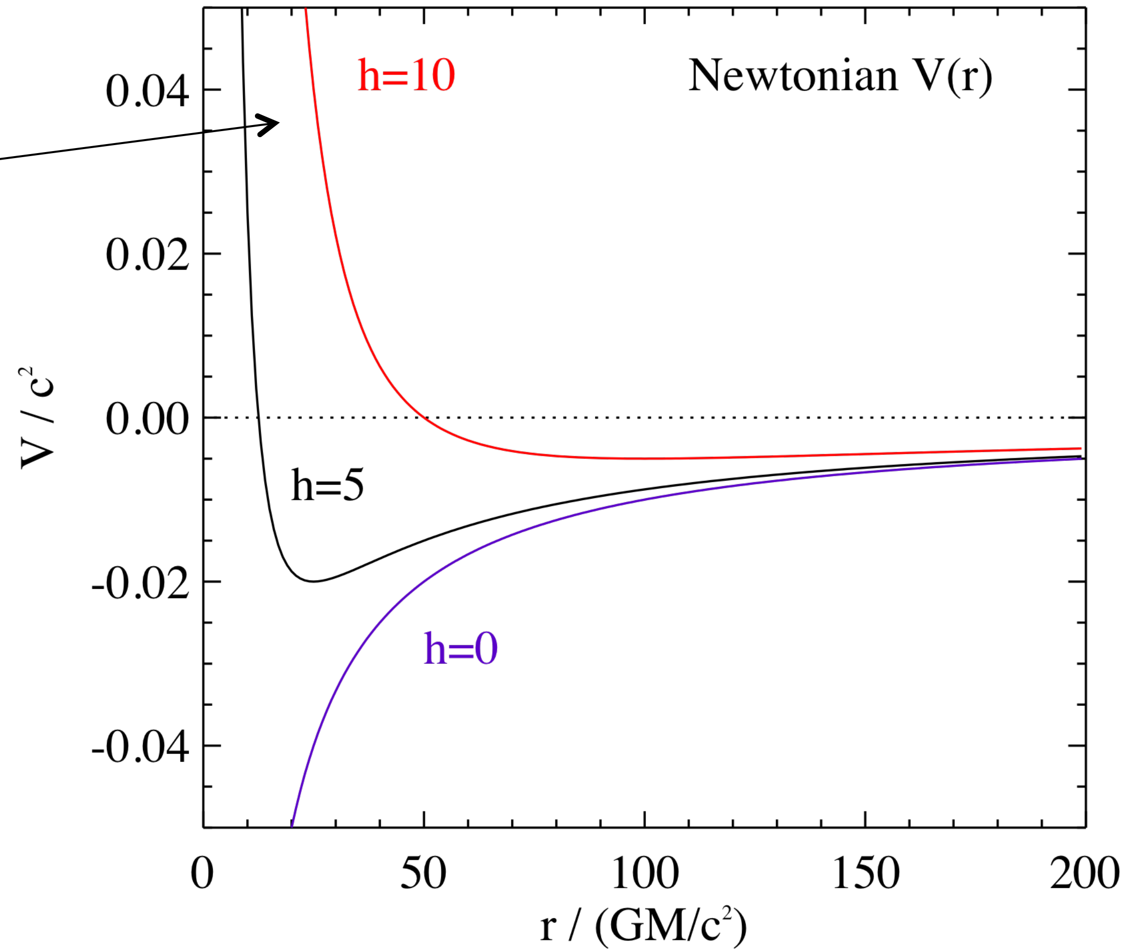
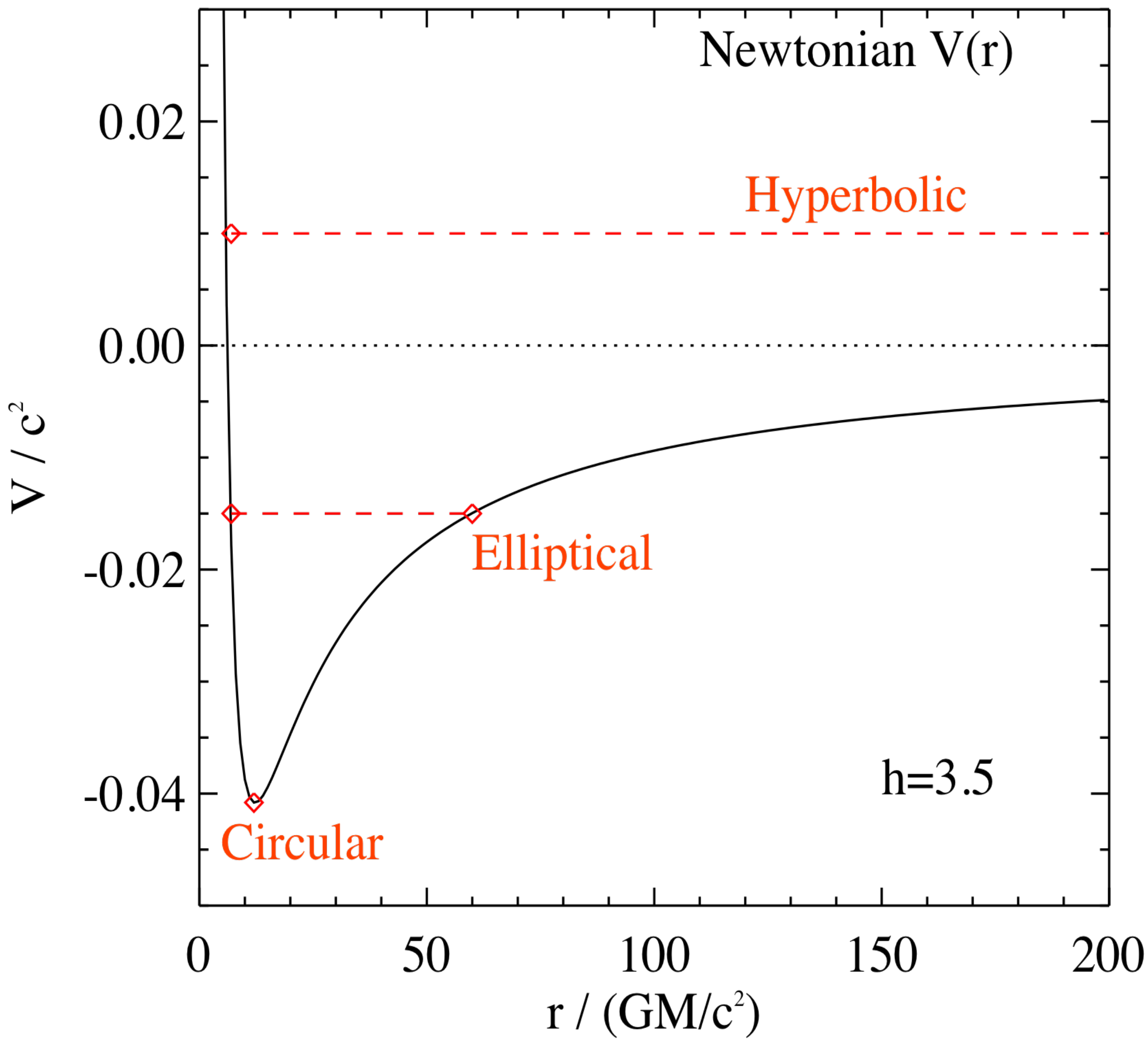
Centrifugal barrier



Orbits in the Newtonian Potential



If total energy E > V(r) at a given r, the orbit is allowed, with the excess contributing kinetic energy

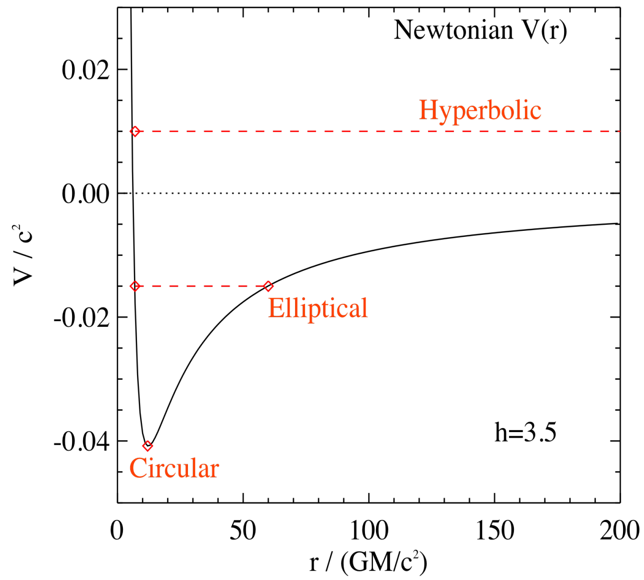
There are three types of allowed orbit in the Newtonian potential

Orbits in the Newtonian Potential

- The centrifugal barrier always dominates as r -> 0
- 2 types of orbits: unbound (hyperbolic) for E>0

  bound (elliptical, circular) for E<0
- Circular: $\dot{r} = 0$, r=rc such that $\ddot{r} = 0$

  $\Rightarrow dV/dr = V'(r)=0$
- Newtonian orbits do not precess



No precession in Newtonian Orbits

To see last point, expand potential around $r = r_C$:

$$V(r) \approx V(r_c) + \frac{1}{2} V''(r_C)(r - r_C)^2.$$

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$$\omega_r^2 = V''(r_C).$$

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Given the Newtonian effective potential

$$V(r) = \frac{h^2}{2r^2} - \frac{GM}{r},$$

so

$$V'(r) = \frac{-h^2}{r^3} + \frac{GM}{r^2}.$$

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$V'(r_C) = 0 \implies h^2 = GMr_C$, therefore

$$V''(r_C) = \frac{3h^2}{r_C^4} - \frac{2GM}{r_C^3} = \frac{GM}{r_C^3}.$$

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$$V''(r_C) = \frac{3h^2}{r_C^4} - \frac{2GM}{r_C^3} = \frac{GM}{r_C^3}.$$

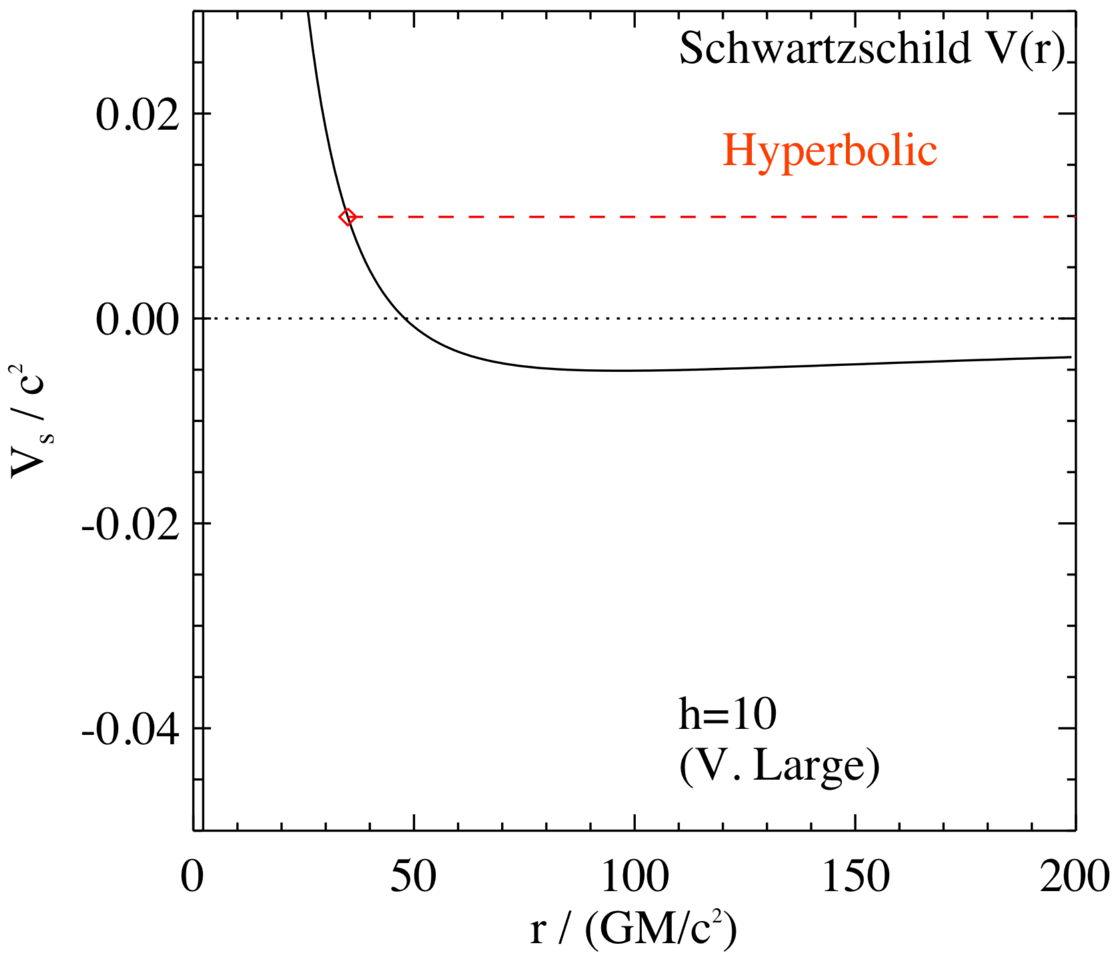
However, $\omega_\phi^2 = GM/r_C^3$, thus $\omega_r = \omega_\phi$. $\implies$ always reach minimum r at same ϕ, so <u>no precession</u>.

Behaviour of the Schwartzschild Potential

$$V(r) = \frac{h^2}{2r^2}\left(1 - \frac{2\mu}{r}\right) - \frac{\mu c^2}{r}$$

Extra term in $-1/r^3$ causes a drop in potential at small r

(This dominates close to the Schwartzschild limit)



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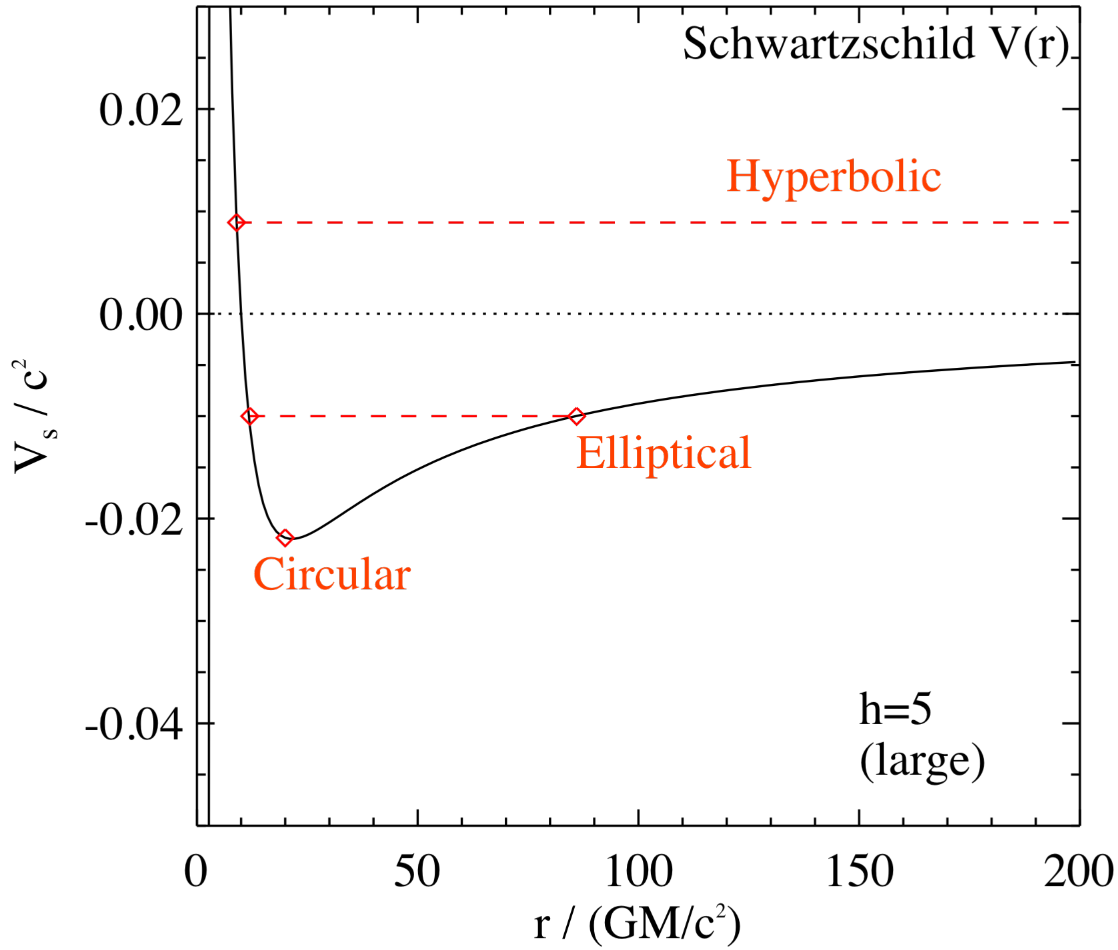
At large h the behaviour is otherwise much like the Newtonian case



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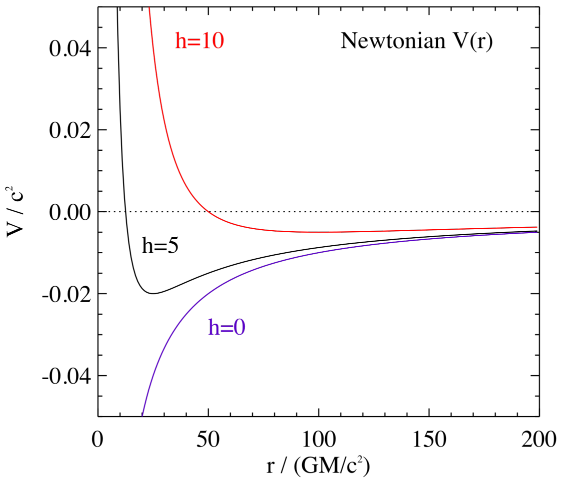
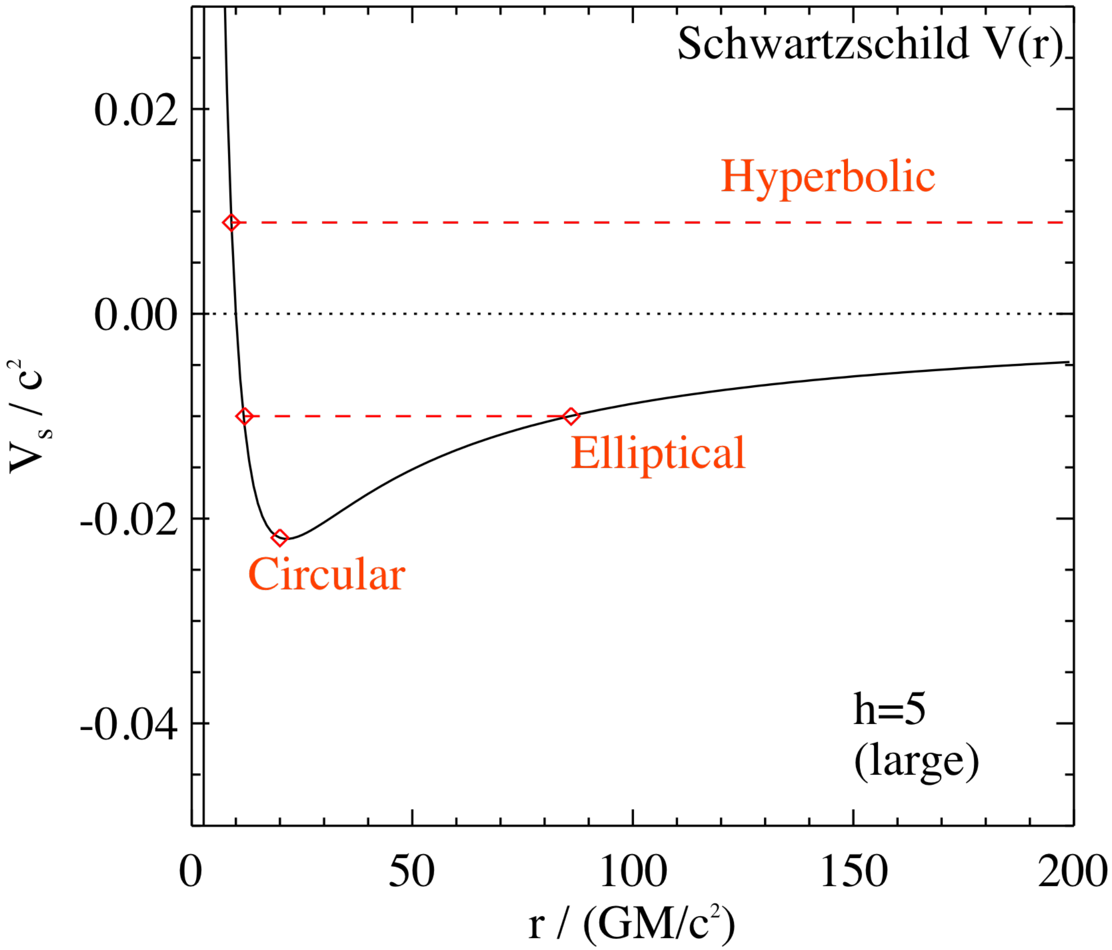


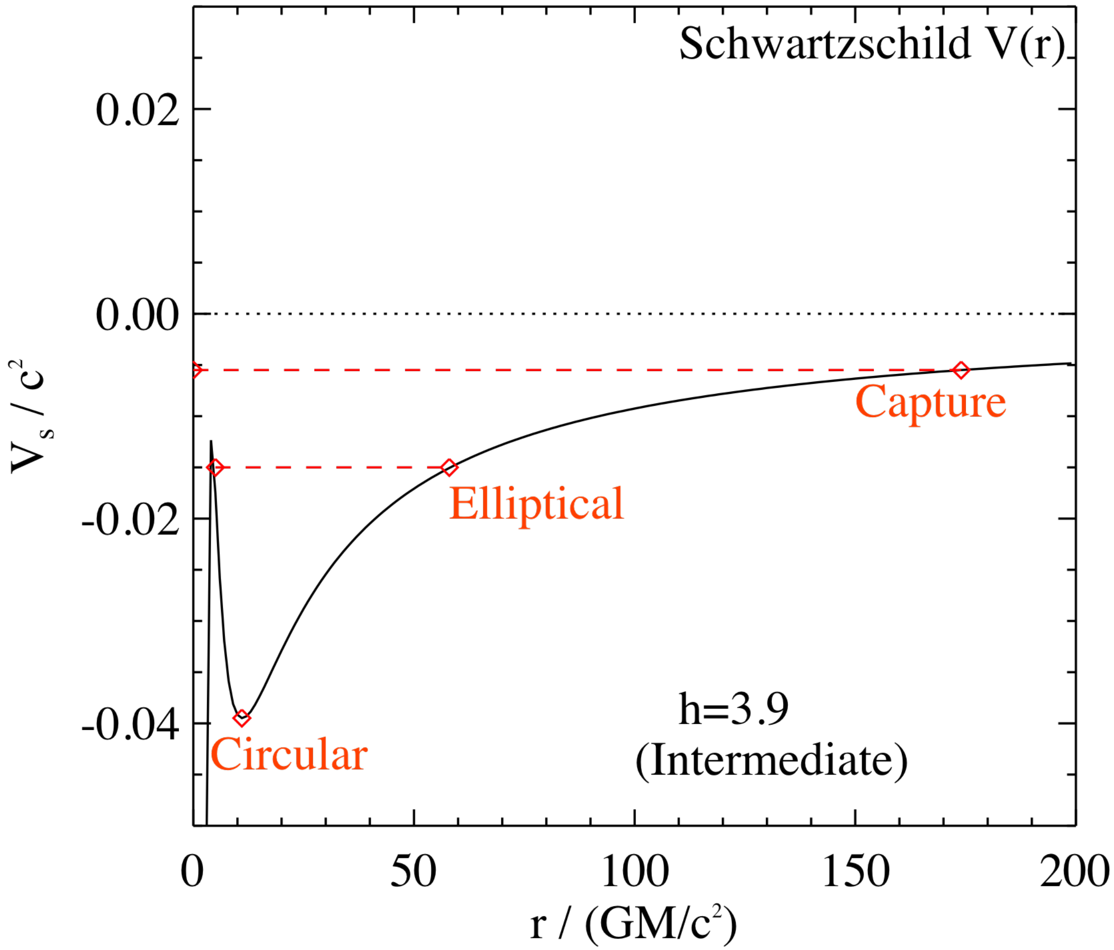


Behaviour of the Schwartzschild Potential

$$V(r) = \frac{h^2}{2r^2}\left(1 - \frac{2\mu}{r}\right) - \frac{\mu c^2}{r}$$

At intermediate h we have a new type of orbit:

capture orbits lead directly onto the central object



Behaviour of the Schwartzschild Potential

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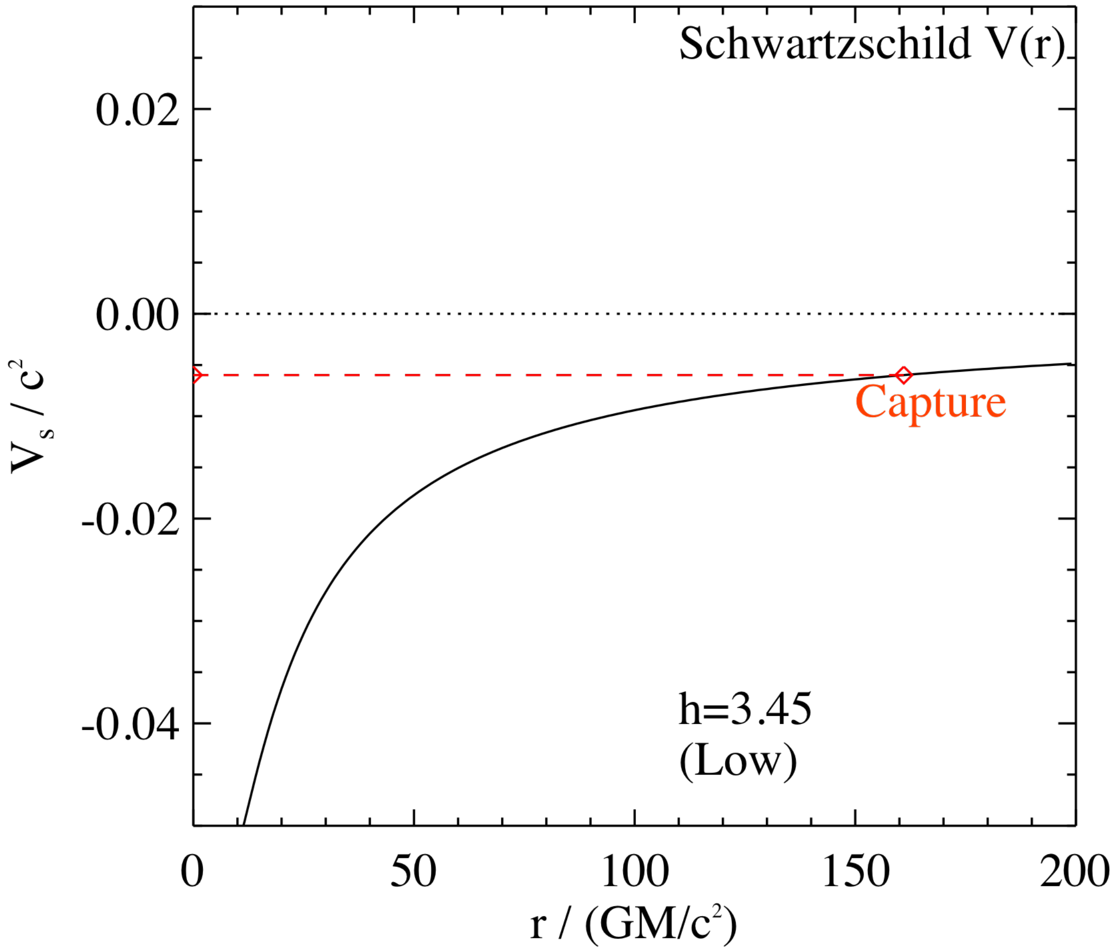
At low h, the gravitational term dominates and only capture orbits are permitted.

There are no bound orbits



Instability of Circular Orbits

The Schwarzschild effective potential is

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At the radius of circular orbits, $dV(r)/dr = V'(r) = 0 \implies$

$$V'(r) = -\frac{h^2}{r^3} + \frac{3h^2\mu}{r^4} + \frac{\mu c^2}{r^2} = 0,$$

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so

$$r_C = \frac{h^2 \pm \sqrt{h^4 - 12h^2\mu^2c^2}}{2\mu c^2}.$$

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In accretion discs around non-rotating black-holes no more energy is available from within this radius. Calculate energy lost using $E = kmc^2$.

Instability of Circular Orbits

Since $\dot{r} = 0$, $r = 6\mu$ and $h^2 = 12\mu^2 c^2$:

$$c^2(k_C^2 - 1) \;=\; \frac{h^2}{r^2}\left(1 - \frac{2\mu}{r}\right) - \frac{2\mu c^2}{r},$$

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$$\begin{aligned} c^2(k_C^2 - 1) &= \frac{h^2}{r^2}\left(1 - \frac{2\mu}{r}\right) - \frac{2\mu c^2}{r}, \\ &= \frac{12\mu^2 c^2}{36\mu^2}\left(1 - \frac{2}{6}\right) - \frac{2c^2}{6}, \\ &= -\frac{1}{9}c^2. \end{aligned}$$

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Thus $k_C^2 = 8/9$. A mass dropped from rest at $r = \infty$ starts with $k = 1$, and thus $1 - k_C = 5.7\%$ of the rest mass must be lost to radiation.

Power from Accretion

- Accretion onto black holes (central, massive singularities) has been proposed as an energy source
  - Schwartzschild accretion from a circular orbit at $3R_s$ yields 5.7% of the rest mass energy
  - The equivalent Newtonian value is $GM/6R_s = 1/12 =$ 8.3%
  - Accretion onto a *rotating* black hole (which obeys the Kerr metric) can yield 42%
  - H-> He Fusion yields 0.7%

Precession in Schwartzschild Geometry

- We looked at precession in Newtonian orbits (where there is none)
- Now consider the Schwartzschild potential:

As for Newton, oscillations in r occur at $\omega_r^2 = V''(r_c)$ but now, setting $\mu = GM/c^2$,

$$V(r) = \frac{h^2}{2r^2}\left(1 - \frac{2\mu}{r}\right) - \frac{\mu c^2}{r} = h^2\left(\frac{1}{2r^2} - \frac{\mu}{r^3}\right) - \frac{\mu c^2}{r}.$$

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First obtain a condition on h for circular orbits of radius r from $V'(r) = 0$:

$$V'(r) = h^2\left(-\frac{1}{r^3} + \frac{3\mu}{r^4}\right) + \frac{\mu c^2}{r^2} = 0,$$

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$$h^2 = \frac{\mu c^2 r^2}{r - 3\mu}.$$

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$$V''(r) = h^2\left(\frac{3}{r^4} - \frac{12\mu}{r^5}\right) - \frac{2\mu c^2}{r^3},$$

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&= \frac{\mu c^2}{r^3(r - 3\mu)}\left(3r - 12\mu - 2(r - 3\mu)\right), \\
&= \frac{\mu c^2 (r - 6\mu)}{r^3(r - 3\mu)}.
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$$

Thus

$$
\omega_r^2 = \left(\frac{r - 6\mu}{r - 3\mu}\right)\frac{\mu c^2}{r^3}.
$$

NB $\omega_r^2 \to 0$ as $r \to 6\mu = 6GM/c^2$ as expected for the last circular orbit.

Precession in Schwartzschild Geometry

Therefore successive close approaches to the star (periastron) occur on a period of

$$P_r = \frac{2\pi}{\omega_r},$$

measured in terms of the proper time of the orbiting particle.

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Therefore, subtracting 2π, the periastron precesses by an amount

$$\Delta\phi \;=\; 2\pi\left[\frac{1}{r^2}\left(\frac{\mu c^2 r^2}{r-3\mu}\right)^{1/2}\left(\frac{r-3\mu}{r-6\mu}\right)^{1/2}\left(\frac{r^3}{\mu c^2}\right)^{1/2} - 1\right],$$

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If $r \gg \mu$ this can be approximated as $\delta\phi \approx 6\pi\mu/r$ rads/orbit, or

$$\boxed{\delta\phi \approx \frac{6\pi GM}{c^2 r}\ \text{rads/orbit.}}$$

The precession is in the direction of the orbit (prograde).

Next Lecture

- Precession of the Perihelion of Mercury

- Equations of Motion for Photons