HOMEWORK 5 SOLUTIONS.

1) $$f(\theta,\phi) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1)\, e^{i\delta_l} \sin\delta_l \, P_l(\cos\theta)$$

$$\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2$$

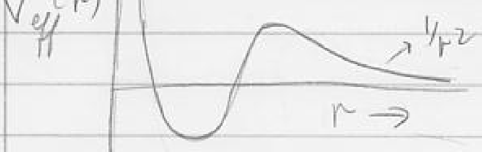
$$\Rightarrow \sigma = \int |f|^2 d\Omega = \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi \, |f(\theta,\phi)|^2$$

(FROM LECTURE NOTES)

$$\sigma = \iint \frac{1}{k^2} \sum_{l,l'} (2l+1)(2l'+1) \sin\delta_l \sin\delta_{l'}\, e^{-i\delta_l} e^{i\delta_{l'}} \cdot \underbrace{P_l(\cos\theta)\, P_{l'}(\cos\theta) \sin\theta \, d\theta}_{\substack{\text{by orthogonality of Legendre} \\ \text{Polynomials, } \theta \text{ integral} \\ = \frac{2}{(2l+1)}\delta_{ll'}}} \;\; \underbrace{d\phi}_{\substack{\text{Integral} \\ \text{over } \phi \\ = 2\pi}}$$

$$\sigma = \frac{4\pi}{k^2} \sum_l (2l+1) \sin^2\delta_l$$

Now, each partial wave (except the $l=0$ partial wave) experiences a <u>centrifugal barrier</u> resulting from the repulsive centrifugal potential term $\frac{l(l+1)}{r^2}$



$$V_{eff}(r) = U(r) + \frac{l(l+1)}{r^2}$$

Only if the particle has sufficient energy to penetrate the barrier will it sample the finite range potential $U(r)$ which is responsible for the phase-shift δ_ℓ.

$l=0$ has no barrier $\left(\frac{\ell(\ell+1)}{r^2} = 0 \right)$ so in the low energy limit only the $l=0$ partial wave samples $r \lesssim a$ (where $r = 0 - a$ is the range of $U(r)$. In effect $\delta_\ell \propto k^{(2\ell+1)}$.

Hence $\delta_\ell = 0$ unless $l=0$ and

$$\sigma = \frac{4\pi}{k^2} \sin^2 \delta_0 .$$

2) $f(\theta,\phi) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) \sin\delta_l \, e^{i\delta_l} P_l(\cos\theta)$ ①

$P_0(\cos\theta) = 1$, $P_1(\cos\theta) = \cos\theta$,

$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$

ie P_0 is a polynomial of order $(\cos\theta)^0$

P_1 " " " $(\cos\theta)^1$

P_2 " " " $(\cos\theta)^2$ etc

now $\frac{d\sigma}{d\Omega} = |f|^2$. so if we summed in ① up to $P_2(\cos\theta)$ (ie $l=2$) we would have a quartic in $\cos\theta$, ie $[3\cos^2\theta - 1]^2$ gives a $\cos^4\theta$ term. So only $l = 0, 1$ are important. Hence, from ①, sum $l=0$, $l=1$

$$f(\theta,\phi) \simeq \frac{1}{k}\left[\sin\delta_0 \, e^{i\delta_0} + 3\sin\delta_1 \, e^{i\delta_1}\cos\theta\right]$$

$$|f|^2 = f \cdot f^* = \frac{1}{k^2}\left[\sin^2\delta_0 + 9\sin^2\delta_1\cos^2\theta + 6\sin\delta_0\sin\delta_1\cos(\delta_1 - \delta_0)\cos\theta\right]$$

Hence:

$\sin^2\delta_0 = .117 \Rightarrow \delta_0 = 20^\circ$ $(\sin\delta_0 = .342)$

$9\sin^2\delta_1 = .011 \Rightarrow \delta_1 = 2^\circ$ $(\sin\delta_1 = .035)$

check that $6\sin\delta_0\sin\delta_1\cos(\delta_1 - \delta_0) = .0681$ ✓