QM4226
QUANTUM THEORY &
HOMEWORK PROBLEMS 4

These questions form part of the continuous assessment of the course. To be handed in by ...

1. Construct the state: $|j_1 j_2 JM> = |3/2\ 1\ 5/2\ 1/2>$ using the eigenstates $|j_1 m_1>|j_2 m_2>$, of the one-particle operators, using the tables of Clebsch-Gordan coefficients.
2. Repeat for $|1/2\ 1\ 3/2\ 1/2>$.
3. Work out $< J_{1Z} >$ for the states in (1) and (2).
4. A quantum state is in a superposition of the orthonormal eigenstates: $\chi = \sqrt{\frac{2}{3}}|3/2\ 1\ 5/2\ 1/2> + \sqrt{\frac{1}{3}}|1/2\ 1\ 3/2\ 1/2>$. Work out $< \mathbf{J_1}.\mathbf{J_2} >$.
5. For the same state, work out $< J_{1+}J_{2-} + J_{2+}J_{1-} >$. Hint: Use Eqs.(4.74) and (4.75) for the above two questions.
6. Two electrons interact via a potential: $V(\mathbf{r}) = A\boldsymbol{\sigma_1}.\boldsymbol{\sigma_2}$ The electrons are in a orthonormal spatial eigenstate, and a spin singlet state, ie $\Psi(\mathbf{r})\chi = R_{ns}(r)Y_{00}\sqrt{\frac{1}{2}}(\alpha_1\beta_2 - \beta_1\alpha_2)$ Work out the interaction energy $< V(\mathbf{r}) >$.
7. $\mathbf{n} = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$ is a vector of unit length, representing the orientation of the proton-proton axis. Show that:

   $$(\mathbf{S}.\mathbf{n}) = \frac{1}{2}[S_+e^{-i\phi}\sin\theta + S_-e^{i\phi}\sin\theta + 2S_z\cos\theta]$$

   Represent $(\mathbf{S}.\mathbf{n})$ as a matrix, using the eigenstates of $\mathbf{S_z}$. Hence, obtain the eigenstates $|n\pm>$ of $(\mathbf{S}.\mathbf{n})$.

   Work out the expectation value of $(\mathbf{S}.\mathbf{n})^2$ if a quantum system is in the spin-state $\chi = \sqrt{\frac{2}{3}}|\alpha> + \sqrt{\frac{1}{3}}|\beta>$.

ANSWERS:

1) $$|j_1 j_2 J M> = \sum_{\substack{m_1 \\ M-m_1}} <j_1 m_1 j_2 m_2 | j_1 j_2 J M> |j_1 m_1>|j_2 m_2>$$

$$|\tfrac{3}{2}\ 1\ \tfrac{5}{2}\ \tfrac{1}{2}> = \sqrt{\tfrac{1}{10}}\,|\tfrac{3}{2}\ \tfrac{3}{2}>|1\ {-1}> + \sqrt{\tfrac{3}{5}}\,|\tfrac{3}{2}\ \tfrac{1}{2}>|1\ 0> + \sqrt{\tfrac{3}{10}}\,|\tfrac{3}{2}\ {-\tfrac{1}{2}}>|1\ 1>$$

2) $$|\tfrac{1}{2}\ 1\ \tfrac{3}{2}\ \tfrac{1}{2}> = \sqrt{\tfrac{1}{3}}\,|\tfrac{1}{2}\ {-\tfrac{1}{2}}>|1\ 1> + \sqrt{\tfrac{2}{3}}\,|\tfrac{1}{2}\ \tfrac{1}{2}>|1\ 0>$$

(3) $J_{1z}|j_1 j_2 JM\rangle = \sum_{\substack{m_1\\m_2}} \hat{J}_{1z} C_{m_1 m_2}|j_1 m_1\rangle|j_2 m_2\rangle$

and $J_{1z}|j_1 m_1\rangle = m_1\hbar|j_1 m_1\rangle$ hence

$J_{1z}|j_1 j_2 JM\rangle = \sum_{m_1} m_1\hbar C_{m_1 m_2}|j_1 m_1\rangle|j_2 m_2\rangle$

So

$\langle j_1 j_2 JM \; J_{1z} \; j_1 j_2 JM\rangle = \sum_{\substack{m_1\\m_2}} m_1\hbar |C_{m_1 m_2}|^2$,

using the orthogonality of the $|j_1 m_1\rangle|j_2 m_2\rangle$

$\langle \tfrac{3}{2}\,1\,\tfrac{5}{2}\,\tfrac{1}{2} \; J_{1z} \; \tfrac{3}{2}\,1\,\tfrac{5}{2}\,\tfrac{1}{2}\rangle = \frac{1}{10}\cdot\tfrac{3}{2}\hbar + \frac{3}{5}\cdot\tfrac{1}{2}\hbar + \tfrac{3}{10}(-\tfrac{1}{2}\hbar)$

$= \frac{3\hbar}{10}$

and

$\langle \tfrac{1}{2}\,1\,\tfrac{3}{2}\,\tfrac{1}{2} \; J_{1z} \; \tfrac{1}{2}\,1\,\tfrac{3}{2}\,\tfrac{1}{2}\rangle = \tfrac{1}{3}\left(-\frac{\hbar}{2}\right) + \frac{2}{3}\frac{\hbar}{2} = \frac{\hbar}{6}$

(4) $2\,\underline{J}_1\cdot\underline{J}_2 = J^2 - J_1^2 - J_2^2 \quad (4.74)$

$\langle \chi \; \underline{J}_1\cdot J_2 \; \chi\rangle = \frac{2}{3}\langle \tfrac{3}{2}\,1\,\tfrac{5}{2}\,\tfrac{1}{2} \underbrace{\underline{J}_1\cdot\underline{J}_2}_{①} \tfrac{3}{2}\,1\,\tfrac{5}{2}\,\tfrac{1}{2}\rangle$

$+ \tfrac{1}{3}\langle \tfrac{1}{2}\,1\,\tfrac{3}{2}\,\tfrac{1}{2} \underbrace{\underline{J}_1\cdot\underline{J}_2}_{②} \tfrac{1}{2}\,1\,\tfrac{3}{2}\,\tfrac{1}{2}\rangle$

By (4.74), ① becomes

$\tfrac{2}{3}\langle \tfrac{3}{2}\,1\,\tfrac{5}{2}\,\tfrac{1}{2} \,|\, \tfrac{1}{2}[J^2 - J_1^2 - J_2^2]\, \tfrac{3}{2}\,1\,\tfrac{5}{2}\,\tfrac{1}{2}\rangle$

$= \tfrac{2}{3}\times\tfrac{1}{2}[\tfrac{5}{2}(\tfrac{5}{2}+1) - \tfrac{3}{2}(\tfrac{3}{2}+1) - 2]\hbar^2 = \hbar^2$

4) (continued) term (2)

$$\tfrac{1}{3}\langle \tfrac{1}{2}\,|\,\tfrac{3}{2}\,\tfrac{1}{2}\ \underline{J}_1\cdot\underline{J}_2\ \tfrac{1}{2}\,|\,\tfrac{3}{2}\,\tfrac{1}{2}\rangle = \frac{1}{3}\cdot\frac{1}{2}\left[\tfrac{3}{2}(\tfrac{3}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}+1)-2\right]\hbar^2$$

$$= \tfrac{1}{6}\hbar^2$$

hence $\langle x\ \underline{J}_1\cdot\underline{J}_2\ x\rangle = \tfrac{7}{6}\hbar^2$

(5) FROM (4.75)

$$\langle J_{1+}J_{2-}+J_{2+}J_{1-}\rangle = \langle J^2-J_1^2-J_2^2-2J_{1z}J_{2z}\rangle$$

We worked out $\tfrac{1}{2}\langle J^2-J_1^2-J_2^2\rangle$ in q. 6

What we need is $2\langle x\ J_{1z}\cdot J_{2z}\ x\rangle$

$$J_{1z}J_{2z}\sum_{\substack{m_1\\m_2}} C^{JM}_{m_1m_2}|j_1m_1\rangle|j_2m_2\rangle = \sum C^{JM}_{m_1m_2}(m_1\hbar)(m_2\hbar)|j_1m_1\rangle\cdot|j_2m_2\rangle$$

then

$$\langle J_{1z}J_{2z}\rangle = \sum_{\substack{m_1\\m_2=M-m_1}} |C_{m_1m_2}|^2\, m_1m_2\hbar^2$$

So

$$2.\langle \tfrac{3}{2}\,|\,\tfrac{5}{2}\,\tfrac{1}{2}\ J_{1z}J_{2z}\ \tfrac{3}{2}\,|\,\tfrac{5}{2}\,\tfrac{1}{2}\rangle = -\frac{3}{5}\hbar^2$$

and

$$2.\langle \tfrac{1}{2}\,|\,\tfrac{3}{2}\,\tfrac{1}{2}\ J_{1z}J_{2z}\ \tfrac{1}{2}\,|\,\tfrac{3}{2}\,\tfrac{1}{2}\rangle = -\tfrac{2}{3}\hbar^2$$

So $\langle x\,2J_{1z}\cdot J_{2z}\,x\rangle = -\tfrac{2}{3}\cdot\tfrac{3}{5}\hbar^2 - \tfrac{1}{3}\cdot\tfrac{2}{3}\hbar^2 = -2\hbar^2\cdot\frac{23}{90}$

But

$\langle x\ J^2-J_1^2-J_2^2\,x\rangle = \tfrac{7}{3}\hbar^2$ (FROM Q.6)

$$\Rightarrow \langle J_{1+}J_{2-}+J_{2+}J_{1-}\rangle = \frac{7}{3}\hbar^2 + 2\hbar^2\frac{23}{90} = \frac{128}{45}\hbar^2$$

6) Since $\underline{\sigma}$ is related to $\underline{S}$ by a simple $\hbar/2$ factor, can use:

$$\underline{\sigma}_1 \cdot \underline{\sigma}_2 = \tfrac{1}{2}\left[\sigma^2 - \sigma_1^2 - \sigma_2^2\right]$$

This operator acts only on the spin part of $\Psi(\underline{r})$

$$\langle V(\underline{r})\rangle = \langle \psi(\underline{r})\chi\, A\sigma_1\cdot\sigma_2\, \psi(\underline{r})\chi\rangle$$

$$= \underbrace{\langle \psi(\underline{r})\,\psi(\underline{r})\rangle}_{=1} \quad \langle \chi\, A\, \underline{\sigma}_1\cdot\underline{\sigma}_2\, \chi\rangle$$

since $R\, Y_{\ell m}(\theta,\phi)$ normalised

$$\langle \chi\, \sigma_1\sigma_2\, \chi\rangle = \tfrac{1}{2}\underbrace{\langle 00\, \sigma^2\, 00\rangle}_{=0} - \tfrac{1}{2}\underbrace{\langle 00\, \sigma_1^2\, 00\rangle}_{3/2} - \tfrac{1}{2}\underbrace{\langle 00\, \sigma_2^2\, 00\rangle}_{3/2}$$

since total spin = 0
for this state

hence $A\langle \chi\, \sigma_1\sigma_2\, \chi\rangle = -3A$

7)

$$\underline{S}\cdot\hat{\underline{n}} = S_x \cos\phi\sin\theta + S_y \sin\phi\sin\theta + S_z\cos\theta$$

Using $S_x = \dfrac{S_+ + S_-}{2}$ and $S_y = \dfrac{S_+ - S_-}{2i}$

and substituting

$$\underline{S}\cdot\underline{n} = \tfrac{1}{2}(S_+ + S_-)\cos\phi\sin\theta + \frac{1}{2i}(S_+ - S_-)\sin\phi\sin\theta$$
$$+ 2S_z\cos\theta$$

$$= \tfrac{1}{2}\left[S_+\sin\theta(\cos\phi - i\sin\phi) + S_-\sin\theta(\cos\phi + i\sin\phi) + 2S_z\cos\theta\right]$$

$$= \tfrac{1}{2}\left[S_+ e^{-i\phi}\sin\theta + S_- e^{i\phi}\sin\theta + 2S_z\cos\theta\right]$$

7) $$\underline{S}\cdot\underline{n} = \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{bmatrix}$$

(given in lecture notes)

$$|n+\rangle = \begin{bmatrix} \cos\theta/2 \; e^{-i\phi/2} \\ \sin\theta/2 \; e^{i\phi/2} \end{bmatrix}$$

$$|n-\rangle = \begin{bmatrix} -\sin\theta/2 \; e^{-i\phi/2} \\ \cos\theta/2 \; e^{i\phi/2} \end{bmatrix}$$

For a spin-½ state, any component $\langle S_x^2\rangle, \langle S_y^2\rangle \ldots \langle S_n^2\rangle = \hbar^2/4$ along an arbitrary direction.

For full marks this should be proved explicitly by evaluating $\langle \chi\, S_n^2\, \chi\rangle = \hbar^2/4$