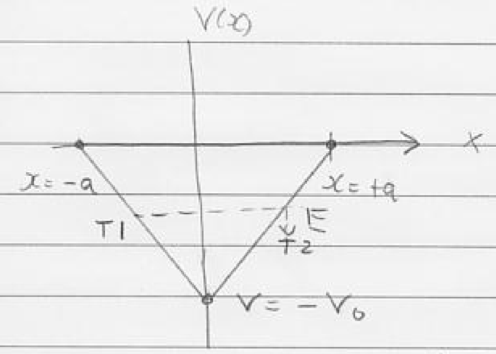
HOMEWORK 3 SOLUTIONS

(1)



FOR TURNING POINTS
$E - V(x) = 0$

here at
$$x = \pm a\left(\frac{E+V_0}{V_0}\right) = \pm T$$

There are no rigid walls, for which $V = \infty$.

Hence
$$\int_{-T}^{+T} k(x)\,dx = (n+1/2)\pi$$

[3]
$$= 2\int_0^T \left\{\frac{2m}{\hbar^2}\left(E+V_0-\frac{V_0}{a}|x|\right)\right\}^{1/2} dx \quad \text{since}$$

$k(x) = \sqrt{\frac{2m}{\hbar^2}(E-V(x))}$; The integral is symmetric, hence

$$(n+1/2)\pi = 2\cdot\sqrt{\frac{2m}{\hbar^2}}\,(E+V_0)^{1/2}\int_0^T\left(1-\frac{V_0 x}{a(E+V_0)}\right)^{1/2} dx$$

$$= 2\sqrt{\frac{2m(E+V_0)}{\hbar^2}}\underbrace{\int_0^T\left(1-\frac{x}{T}\right)^{1/2} dx}$$

$$= \left[-\frac{2T}{3}\left(1-\frac{x}{T}\right)^{3/2}\right]_0^T = \frac{2T}{3}$$

[3]
$$\Rightarrow (n+1/2)\pi = \frac{4T}{3}\sqrt{\frac{2m(E+V_0)}{\hbar^2}} = \frac{4a}{3}\sqrt{\frac{2m}{\hbar^2}}\,\frac{(E+V_0)^{3/2}}{V_0}$$

since $\frac{m a^2}{\hbar^2} = \frac{50}{V_0}$

$$(n+\tfrac{1}{2})\pi = \frac{4}{3}\sqrt{\frac{100}{V_0}}\left(\frac{E+V_0}{V_0}\right)^{3/2}$$

$$(n+\tfrac{1}{2})\pi = \frac{40}{3}\left(\frac{E+V_0}{V_0}\right)^{3/2}$$

$$E_n = -V_0 + V_0\left[\frac{3}{40}(n+\tfrac{1}{2})\pi\right]^{2/3}$$

For bound state $E_n < 0$

so $\frac{3}{40}(n+\frac{1}{2})\pi < 1$

$n = 0, 1, 2, 3$ are OK

[2] $\frac{E_n}{V_0} = -.76,\ -.50,\ -.30,\ -.12$

2) In WKB approximation, $T = \text{Exp} - 2\lambda$

and $\lambda = \int_a^b q(x)\,dx$

The turning points are $x = \pm \frac{V_0}{E} = \pm b$

$$q(x) = \sqrt{\frac{2m(V(x) - E)}{\hbar^2}} = \sqrt{\frac{2m}{\hbar^2}}\left[\frac{V_0}{|x|} - E\right]^{1/2} = \sqrt{\frac{2mE}{\hbar^2}}\left[\frac{b}{|x|} - 1\right]^{1/2}$$

So

$$\lambda = \int_{-b}^{b} q(x)\,dx = \sqrt{\frac{2mE}{\hbar^2}} \cdot 2 \int_0^b \left[\frac{b}{x} - 1\right]^{1/2} dx = \sqrt{\frac{4mE}{\hbar^2}} \cdot I$$

take $x = b\cos^2\theta$; $I = \int_{\pi/2}^{0} -2b\sin^2\theta\,d\theta = \int_0^{\pi/2} 2b\sin^2\theta\,d\theta$

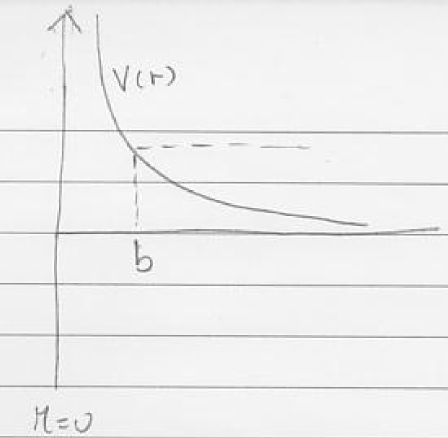
$$= b\int_0^{\pi/2} (1 - \cos 2\theta)\,d\theta = b\pi/2$$

hence $\lambda = \sqrt{\frac{2m\pi^2 E}{\hbar^2}}\left(\frac{V_0}{E}\right) = Cst.\frac{V_0}{E^{1/2}}$

$$T_{WKB} \simeq \text{Exp} - 2\lambda \simeq \text{Exp} - 2Cst.\, V_0/E^{1/2}$$

$$\ln T_{WKB} \propto -\frac{V_0}{E^{1/2}}$$

3)



can consider
<u>two</u> turning points at

$a = 0$

$b = \dfrac{e^2}{4\pi\varepsilon_0 E}$

$T = \text{Exp} - 2\lambda$

$\lambda = \int_a^b q(r)\, dr$

$$q(r) = \left\{ \frac{2m}{\hbar^2} \left[\frac{e^2}{4\pi\varepsilon_0 r} - E \right] \right\}^{1/2} = A \left[\frac{1}{r} - \frac{1}{b} \right]^{1/2}$$

$$A = \left(\frac{2m}{\hbar^2} \frac{e^2}{4\pi\varepsilon_0} \right)^{1/2}$$

$$\int_0^b \left[\frac{1}{r} - \frac{1}{b} \right]^{1/2} dr, \text{ with } r = b\cos^2\theta, = b^{1/2} \int_0^{\pi/2} (1 - \cos 2\theta)\, d\theta$$

$$= b^{1/2} \frac{\pi}{2}$$

$$\Rightarrow \lambda = A b^{1/2} \frac{\pi}{2} = \frac{\pi}{2} \frac{e^2}{4\pi\varepsilon_0} \sqrt{\frac{2m}{\hbar^2 E}}$$

$$T \simeq \text{Exp} - \left\{ \pi \frac{e^2}{4\pi\varepsilon_0} \sqrt{\frac{2m}{\hbar^2 E}} \right\}$$