HOMEWORK 2 SOLUTIONS

Q1) (9 MARKS)

We write the solution of the TDSE:

$$i\hbar \frac{\partial \psi}{\partial t} = [H_0 + H'(t)]\,\psi$$

in the form:

$$\psi(t) = \sum_n c_n(t)\,\phi_n\, e^{-iE_n t/\hbar}$$

(we don't specify spatial coordinates in this case, so I leave them out)

We can obtain coupled equations like eq (2.13) in the notes:

$$\dot{c}_n = \frac{1}{i\hbar}\sum_m H'_{nm}(t)\, c_m(t)\, e^{i\omega_{nm}t}$$

in this case $m, n \equiv a$ or b so

$$\dot{c}_a = \frac{1}{i\hbar}\left[\tilde{H}'_{aa}(t)\, c_a(t) + \tilde{H}'_{ab}(t)\, c_b(t)\right]$$

[3]

and $$\dot{c}_b = \frac{1}{i\hbar}\left[\tilde{H}'_{bb}(t)\, c_b(t) + \tilde{H}'_{ba}(t)\, c_a(t)\right]$$

where $\tilde{H}'_{ab} \equiv H'_{ab}\cdot e^{i\omega_{ab}t}$

So, we just need to work out:

$$\omega_{ab},\ \omega_{ba}$$

$$H'_{aa},\ H'_{bb},\ H'_{ab},\ H'_{ba}$$

$$\omega_{ab} = \frac{E_a - E_b}{\hbar} = -\omega_0$$

$$\omega_{ba} = +\omega_0$$

$$H'_{aa} = \int_{\substack{\text{all}\\\text{space}}} \psi_a^* H'(t)\, \psi_b \, d\tau = \underbrace{\int \psi_a^* (P\cos\omega t - Q\sin\omega t)\psi_b \, d\tau}_{\text{integral over spatial coordinates}}$$

$$= \int \psi_a^* \left(\frac{\hbar\omega_r}{2} \cos\omega t - i\frac{\hbar\omega_r}{2}\sin\omega t \right) \psi_b \, d\tau$$

$$= 0 \quad \text{since} \quad \int \psi_a^* \psi_b \, d\tau = 0$$

(the states ψ_a, ψ_b are orthogonal)

$$H'_{ab} = \int \psi_a^* (P\cos\omega t - Q\sin\omega t)\, \psi_b \, d\tau$$

$$= (\cos\omega t + i\sin\omega t)\frac{\hbar\omega_r}{2} \int \psi_a^* \psi_a \, d\tau$$

$$= \frac{\hbar\omega_r}{2} e^{i\omega t}$$

Similarly, $H'_{bb} = 0$, $H'_{ba} = \frac{\hbar\omega_r}{2} e^{-i\omega t}$ [2]

COUPLED EQUATIONS :

$$\dot{C}_a(t) = \frac{\omega_R}{2i} e^{i(\omega - \omega_0)t} C_b(t)$$

$$\dot{C}_b(t) = \frac{\omega_R}{2i} e^{i(\omega_0 - \omega)t} C_a(t)$$ [4]

HOMEWORK 2.

$\boxed{\text{SOLUTIONS}}$

2) The probability that a system initially in state i, ends up in state k is $|C_k|^2$ where

$$C_{ki}(t) = \frac{1}{i\hbar}\int_{t_0}^{t} \langle k\, H'\, i\rangle\, e^{i\omega_{ki}t'}\, dt'$$

The initial state $\psi_{i=1} = 1/\sqrt{a}\, \cos\frac{\pi x}{2a}$

The matrix elements $\langle k\, H\, 1\rangle$

$$H_{ki} = \left[\frac{A}{\sqrt{a}}\int_{-a}^{a} \psi_k^*(x)\, \underbrace{\sin\frac{5\pi x}{2a}\, \cos\frac{\pi x}{2a}}\, dx\right] e^{-\gamma t}$$

$$= \tfrac{1}{2}\left[\sin\frac{6\pi x}{2a} + \sin\frac{4\pi x}{2a}\right] = \frac{\sqrt{a}}{2}\left[\psi_6 + \psi_4\right]$$

$$H_{ki} = \frac{A}{2}\int_{-a}^{a} \psi_k^*(x)\left[\psi_6(x) + \psi_4(x)\right] dx$$

[3] $= \frac{A}{2}\left[\delta_{k6} + \delta_{k4}\right]$ by orthonormality.

Then,

$$C_k(t=\infty) = \frac{1}{i\hbar}\int_0^{\infty} \frac{A}{2}\left[\delta_{k6} + \delta_{k4}\right] e^{-\gamma t'}\, e^{i\omega_{k1}t'}\, dt'$$

$$= \frac{A}{2i\hbar}\left[\frac{e^{(i\omega_{k1}-\gamma)t'}}{(i\omega_{k1}-\gamma)}\right]_0^\infty (\delta_{k4}+\delta_{k6})$$

$$C_k(\infty) = \frac{A}{2i\hbar}[\delta_{k6}+\delta_{k4}]\left\{\frac{1}{-(i\omega-\gamma)}\right\}$$

$$= \frac{A\,[\delta_{k6}+\delta_{k4}]}{2\hbar(\omega_{k1}+i\gamma)}$$

The angular frequencies are

$$\omega_{k1} = \frac{E_k - E_1}{\hbar} = \frac{\pi^2\hbar}{8ma^2}(k^2-1)$$

$$\omega_{41} = \frac{15\pi^2\hbar}{8ma^2}, \quad \omega_{61} = \frac{35}{8}\frac{\pi^2\hbar}{ma^2}.$$

The probability of ending in $k=4$ state is

$$P_{41} = \frac{|A|^2}{4\hbar^2}\left[\frac{1}{\left(\gamma^2+\left(\frac{15\pi^2\hbar}{8ma^2}\right)^2\right)^2}\right]$$

$$P_{61} = \frac{|A|^2}{4\hbar^2}\left[\frac{1}{\left(\gamma^2+\left(\frac{35\pi^2\hbar}{8ma^2}\right)^2\right)^2}\right]$$

(3) $$C_{k0}(t) = \frac{A}{i\hbar}\int_0^t \langle k\, x^3\, 0\rangle\, e^{i\omega_{k0}t'} \sin\Omega t'\, dt'$$

$$\omega_{k0} = \frac{E_k - E_0}{\hbar}$$

$$= \left[(k+\tfrac{1}{2}) - \tfrac{1}{2}\right]\omega = k\omega$$

$$C_{k0}(t') = \frac{-A}{2\hbar}\int_0^t \langle k\, x^3\, 0\rangle \left[e^{i(k\omega+\Omega)t'} - e^{i(k\omega-\Omega)t'}\right] dt'$$

$$C_{k0}(t) = -A\frac{\langle k\, x^3\, 0\rangle}{2\hbar}\left[\frac{e^{i(k\omega+\Omega)t'}}{i(k\omega+\Omega)} - \frac{e^{i(k\omega-\Omega)t'}}{i(k\omega-\Omega)}\right]_0^t$$

$$\frac{-A}{2i\hbar}\left(\frac{\hbar}{2m\omega}\right)^{3/2}\left\{3\delta_{k1} + \sqrt{6}\,\delta_{k,3}\right\}\left[\frac{e^{i(k\omega+\Omega)t} - 1}{(\Omega + k\omega)} + \frac{e^{i(k\omega-\Omega)t} - 1}{(\Omega - k\omega)}\right]$$

(b) $|C_{k0}|^2$ is large if $\Omega = \pm k\omega$
But from ground state, only absorption term $\Omega = +k\omega$ is useful.

(c) $$P_k(t) = \frac{|A|^2\hbar}{8(m\omega)^3}\,\frac{\sin^2(\Omega - k\omega)\,t/2}{(\Omega - k\omega)^2} \times \begin{cases} 9\,\delta_{k,1} \\ 6\,\delta_{k,3} \end{cases}$$