PHYS4226
QUANTUM THEORY &
HOMEWORK PROBLEMS 2

These questions form part of the continuous assessment of the course. To be handed in by ...

1. Consider an unperturbed system, described by a Hamiltonian H_0, which can only exist in two states a and b, with wave functions ψ_a and ψ_b, corresponding to eigenenergies $E_a = -\hbar\omega_0/2$ and $E_b = \hbar\omega_0/2$, respectively, ω_0 being a constant frequency. The system is acted on by a time-dependent interaction, described by the Hamiltonian,

$$V'(t) = P\cos\omega t - Q\sin\omega t \tag{1}$$

which can induce transitions between the states a and b such that the action of the operators P and Q are:

$$P\psi_a = \frac{\hbar\omega_r}{2}\psi_b;\ \ P\psi_b = \frac{\hbar\omega_r}{2}\psi_a;\ \ Q\psi_a = i\frac{\hbar\omega_r}{2}\psi_b;\ Q\psi_b = -i\frac{\hbar\omega_r}{2}\psi_a. \tag{2}$$

where ω_r is a constant. Write down a set of coupled differential equations for the probability amplitudes $c_a(t)$ and $c_b(t)$ of the two eigenstates ψ_a and ψ_b, explaining all assumptions.

2. A particle of mass m is bound in a one-dimensional potential well, where $V(x) = 0$ for $|x| \leq a$ and $V(x) = \infty$ elsewhere. The orthonormal eigenfunctions ψ_n satisfying $H_0\psi_n = E_n\psi_n$ where H_0 is the Hamiltonian for the system are the even-parity states $\psi_n = \frac{1}{\sqrt{a}}\cos\frac{n\pi x}{2a}$ with $n = 1, 3, 5...$; and the odd-parity states $\psi_n = \frac{1}{\sqrt{a}}\sin\frac{n\pi x}{2a}$ with $n = 2, 4, 6....$ At time $t = 0$ the particle is in its ground-state. It is acted on by a time-dependent perturbation:

$$V(x,t) = A\sin\left(\frac{5\pi x}{2a}\right)\exp -\gamma t$$

where $\gamma > 0$. To which states can the particle be excited, to first order in A?
Calculate the probabilities that as $t \to \infty$, the particle has been excited to each of these states.When evaluating matrix elements, the trig. formulae $2\sin A\cos B = \sin(A+B) + \sin(A-B)$ and $2\sin A\sin B = \cos(A-B) - \cos(A+B)$ should be useful.

3. A quantum system is described by the Hamiltonian $H(\mathbf{r}, t) = H_0(\mathbf{r}) + \lambda V(\mathbf{r}, t)$ where λ is a small real parameter. The eigenfunctions of the unperturbed Hamiltonian H_0 are ϕ_n and E_n respectively. A particle, for $t \leq 0$ is in the ground state $|0>$ of a one-dimensional harmonic oscillator potential of angular frequency ω. At $t = 0$ a perturbation $\lambda V(x,t) = Ax^3\sin\Omega t$, where Ω and A are constants, is turned on.

   (a) Obtain an expression for the transition amplitude $c_k(t)$, for excitation at a later time t to a state ϕ_k, in terms of the matrix elements $< k|x^3|0 >$.

   (b) Show that for Ω near a certain frequency, the probability of finding the system in an excited state is large.

   (c) Evaluate the leading term in this probability for either of the excited states.

   NB Use the result $< k|x^3|0 >= \left(\frac{\hbar}{2m\omega}\right)^{3/2}\left[3\delta_{k,1} + \sqrt{6}\delta_{k,3}\right]$.