QM4226
QUANTUM THEORY &
HOMEWORK SOLUTIONS 1

1. A bound quantum system has a complete set of orthonormal energy eigenfunctions u_n with non-degenerate eigenvalues E_n. The operator Q corresponds to some other observable and is such that:

$$Qu_1 = u_2; \quad Qu_2 = u_1; \quad Qu_n = 0 \ (\ if \ n \geq 3)$$

Show that $\phi_\pm = u_1 \pm u_2$ and $\phi = u_i, \ i \geq 3$ form the complete set of eigenfunctions of Q and give them in normalised form.

(b) If the observable corresponding to Q is measured and is found to have eigenvalue 1, what is the expectation value of the energy in the resulting state?

SOLUTION:
(a)By substitution, can show that $Q(u_1 \pm u_2) = \lambda_\pm(u_1 \pm u_2)$ where $\lambda_+ = 1$ and $\lambda_- = -1$. However $Qu_i = 0$ so all the other eigenfunctions correspond to eigenvalue zero. Normalisation: By requiring that $\int \phi^* \phi d\tau = 1$ we obtain the normalised eigenfunctions $\phi_+ = \frac{1}{\sqrt{2}}(u_1 + u_2)$ and $\phi_- = \frac{1}{\sqrt{2}}(u_1 - u_2)$. Completeness: Any function which may be expressed in terms of the complete set u_i may evidently be expressed in terms of the ϕ_i. You can also use closure relation, ie show by using $\phi_\pm = u_1 \pm u_2$ and $\phi = u_i, \ i \geq 3$ that $\sum_j \phi_j^*(x)\phi_j^*(x') = \sum_i u_i^*(x)u_i^*(x')$, hence if the u_i are complete, then so are the ϕ_j. [3]
(b) The expectation value of the energy for this state is: $< E >= \int \phi_+^* \ H \ \phi_+ d\tau = \int \frac{1}{\sqrt{2}}(u_1^* + u_2^*) \ H \ \frac{1}{\sqrt{2}}(u_1 + u_2)d\tau = \frac{1}{2}(E_1 + E_2)$ since $H \ u_n = E_n u_n$. [2]

2. The operators x and p_x corresponding to position and momentum, respectively, satisfy the fundamental commutator:

$$[x, p_x] = i\hbar$$

if $C_n = [x^n, p_x]$, show that C_n obeys the recurrence relation,

$$C_n = xC_{n-1} + i\hbar x^{n-1}$$

Hence, by repeated application show that:

$$[x^n, p_x] = i\hbar n x^{n-1}$$

You can use $[AB, C] = A[B, C] + [A, C]B$.
SOLUTION: There are several ways of doing this but quickest is to write
$C_n = [x^n, p_x] = [xx^{n-1}, p_x] = x[x^{n-1}, p_x] + [x, p_x]x^{n-1} = xC_{n-1} + i\hbar x^{n-1}$
$= x(xC_{n-2} + i\hbar x^{n-2}) + i\hbar x^{n-1} = x^2 C_{n-2} + 2i\hbar x^{n-1}$ [3]
By repeated applications we obtain,
$C_n = x^{n-1}C_1 + (n-1)i\hbar x^{n-1}$. But $C_1 = i\hbar$ so
$C_n = [x^n, p_x] = i\hbar n x^{n-1}$. [2]

3. By expanding to appropriate order, verify that

$$e^{A+B} \simeq e^A\, e^B\, e^{-[A,B]/2} \simeq e^{A/2}\, e^B\, e^{A/2}$$

where A and B are quantum operators.
SOLUTION This is easy to show by expanding to quadratic order: $\exp A + B = 1 + A + B + \frac{1}{2}(A+B)^2 \simeq 1 + A + B + \frac{1}{2}(A^2 + B^2 + AB + BA)$ while:
$e^A\, e^B\, e^{-[A,B]/2} \simeq (1 + A + \frac{1}{2}A^2)(1 + B + \frac{1}{2}B^2)(1 - \frac{1}{2}(AB - BA))$. Retaining only quadratic powers (or less) we get:
$e^A\, e^B\, e^{-[A,B]/2} \simeq (1 + A + B + AB + \frac{1}{2}A^2 + \frac{1}{2}B^2 - \frac{1}{2}AB + \frac{1}{2}BA) = (1 + A + B + \frac{1}{2}A^2 + \frac{1}{2}B^2 + \frac{1}{2}AB + \frac{1}{2}BA) \simeq \exp A + B$. [3]
Similarly , to quadratic order
$e^{A/2}\, e^B\, e^{A/2} \simeq ((1 + A/2 + \frac{1}{8}A^2)(1 + B + \frac{1}{2}B^2)(1 + A/2 + \frac{1}{8}A^2) \simeq (1 + A + B + \frac{1}{2}(A+B)^2$ if only terms of quadratic order or less are retained. [2]

4. (a) The orthonormal eigenstates $|n>$ of a one-dimensional harmonic oscillator of angular frequency ω and energy $E_n = (n + 1/2)\hbar\omega$, n=0,1,2...may be considered as states containing n quanta each of energy $\hbar\omega$. Consider the normalised state $|\alpha>$ given by:

$$|\alpha> = e^{-|\alpha^2|/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n>$$

Where α is some parameter (generally complex) defining the state. Find:
(a) The probability that the state $|\alpha>$ contains k quanta each of energy $\hbar\omega$.
(b) The average number of quanta in state $|\alpha>$.
NB note that $|\alpha>$ is sometimes termed a *coherent state*. In a position representation it would correspond to the gaussian wavepackets studied in Lec. 2.

SOLUTION The expression:

$$|\alpha> = e^{-|\alpha^2|/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n>$$

Is in the form of a general quantum state, written as a superposition of eigenstates,

$$|\Psi> = \sum_{n=0} C_n |n>$$

where $C_k =< k|\Psi>$; the probability of measuring the system and finding it in state $|k>$ equals $P_k = |C_k|^2$.

Hence for part (a) we see that:

$$P_k = |e^{-|\alpha^2|/2} \frac{\alpha^k}{\sqrt{k!}}|^2$$

$$= e^{-|\alpha^2|} \frac{|\alpha|^{2k}}{k!}$$

For part (b), the average number of quanta is given by:

$$< n >= \sum_{n=0} n \; P_n = \sum_{n=0} n e^{-|\alpha^2|} \; \frac{|\alpha|^{2n}}{n!}$$

$$= \sum_{n=1} n e^{-|\alpha^2|} \; \frac{|\alpha|^{2n}}{n!} = e^{-|\alpha^2|} \sum_{n=1} |\alpha^2| \frac{|\alpha|^{2(n-1)}}{(n-1)!} = e^{-|\alpha^2|} \; |\alpha^2| \sum_{n=0} \frac{|\alpha|^{2n}}{n!} = e^{-|\alpha^2|} \; |\alpha^2| \; e^{+|\alpha^2|} = |\alpha^2|$$

alternatively (for students who did 3rd year quantum course at UCL) can evaluate the expectation value of the number operator, $N = aa^\dagger$, using $N|n >= n|n >$. Then, evaluate $< \alpha|N|\alpha >$ to obtain the result above.