QM4226
QUANTUM THEORY

SUPPLEMENTARY PROBLEMS 1

1. **Revision of quantum probabilities** Consider an electron in a hydrogen atom, described by a wavefunction:

$$\Psi(\mathbf{r}, \mathbf{t}=\mathbf{0}) \propto [\psi_{100} - 2\psi_{210} + 3\psi_{32-2} - 4\psi_{411}]$$

where the ψ_{nlm} are eigenfunctions of the hydrogen atom. Evaluate:

   (a) The normalised wavefunction.

   (b) The probability that the electron is measured to be in the ground state.

   (c) The *expectation* value of the energy. (using $E_n = -1/n^2$ au).

   (d) The expectation value of L_z

   (e) The probability that the electron is measured to have magnitude of L^2 equal to $2\hbar^2$. (remember that the total angular momentum $= l(l+1)\hbar^2$.

   (f) Hence, what is the overall probability of measuring $l = 1$ followed by $m = 0$?

   (g) If the measurements were made in opposite order, what is the probability of measuring $m = 0$, followed by $l = 1$?

2. For the gaussian wavepacket in a box, given in eqs(1.23-1.25). Calculate C_1 and C_2 if $x_0 = 0$.

3. On C.Stroud's website (web address in the notes), an animation is shown, of the time-evolution of a gaussian wavepacket in a box with sides at $x = 0$ and $x = L$. At time $t = 0$ the gaussian was prepared in the centre of the box. Explain why the first revival is observed at $t_R = \frac{mL^2}{2\pi\hbar}$ (Hint: consider symmetry: not all states can contribute to Stroud's wavepacket since it starts in the centre of the box).

4. When would you expect the first revival of a wavepacket moving in a harmonic oscillator potential? ($E_n = (n + 1/2)\hbar\omega$).

5. **Revision** Show that our gaussian wavepacket in a box satisfies the Heisenberg Uncertainty Principle. Work out $< x^2 >$, $< x >$, $< p >$,$< p^2 >$. Hence work out $(\Delta p)^2 =< (p- < p >)^2 >=< p^2 > + < p >^2 -2 < p >< p >$. Then work out $\Delta p \Delta x$. The integrals$_{-\infty}^{\infty} \int x^n \exp -ax^2 dx = \frac{1}{2a}\sqrt{\pi/a}$ if $n = 2$ and zero for $n = 1$.

6. By applying the free particle time-evolution operator to (1.32a) show that at later time, our gaussian wavepacket which initially was $\psi(x, t = 0) = \frac{1}{(2\pi\sigma^2)^{1/4}} \exp -x^2/(4\sigma^2)$, now becomes:

$$\psi(x,t) = \frac{1}{(2\pi\hbar)^{1/2}} \int dp \Phi(p, t=0) \exp \frac{i}{\hbar}[px - \frac{p^2 t}{2m}]$$

7. Since $\Phi(p, t = 0) = FT^{-1}[\psi(x, t = 0)] = A \exp -p^2\sigma^2/\hbar^2$, with $A = \frac{(8\pi\sigma^2)^{1/4}}{\sqrt{2\pi\hbar}}$, evaluate the gaussian integral in the previous question, to prove eq(1.30) and show that the free wavepacket spreads continuously with time. Use $_{-\infty}^{\infty} \int \exp(-ax^2 - bx) dx = \sqrt{\pi/a} \exp \frac{b^2}{4a}$