9.1. Derive the fluid equation starting from the acceleration and Friedmann equations.

The acceleration equation is

$$\ddot{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) R,$$

while the Friedmann equation is

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - kc^2.$$

Taking the derivative of the Friedmann equation

$$2\dot{R}\ddot{R} = \frac{8\pi G}{3}\left(\dot{\rho}R^2 + 2\rho R\dot{R}\right).$$

Substituting for $\ddot{R}$ from the acceleration equation:

$$-\frac{8\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) R\dot{R} = \frac{8\pi G}{3}\left(\dot{\rho}R^2 + 2\rho R\dot{R}\right),$$

hence

$$-\left(\rho + \frac{3p}{c^2}\right) R\dot{R} = \dot{\rho}R^2 + 2\rho R\dot{R}.$$

Dividing through by R^2,

$$\dot{\rho} + 3\frac{\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right) = 0,$$

which is the fluid equation.

9.2. Integrate the Friedmann equation for a flat, matter-only universe (Einstein-de Sitter model) to show that $R \propto t^{2/3}$.

Flat implies $k = 0$ while matter-only means that $\rho \propto R^{-3}$, so the Friedmann equation shows that

$$\dot{R}^2 \propto R^{-1}.$$

Therefore

$$\int R^{1/2}\, dR \propto t,$$

which leads to $R \propto t^{2/3}$.

Hence show that the age of such a universe is given by

$$t_0 = \frac{2}{3} t_H,$$

where $t_H = H_0^{-1}$ is the "Hubble time" and $H_0 = H(t_0)$ is Hubble's constant at time t_0.

Since $R \propto t^{2/3}$, we can write

$$R = R_0 \left(\frac{t}{t_0}\right)^{2/3}.$$

Since $H = \dot{R}/R$, take the derivative

$$\dot{R} = \frac{2}{3t} R_0 \left(\frac{t}{t_0}\right)^{2/3},$$

giving

$$H = \frac{2}{3t},$$

from which the result follows immediately.

9.3. Find the corresponding results to the previous question for a flat, radiation-only universe for which $\rho_R \propto R^{-4}$. (This describes the early universe.)

For $\rho \propto R^{-3}$ and a flat Universe, the Friedmann equation gives

$$\dot{R}^2 \propto R^{-2},$$

and carrying through the same calculation as before gives $R \propto t^{1/2}$. and

$$t = \frac{1}{2H}.$$

9.4. Use the fluid equation to show that a fluid for which $p = -\rho c^2$ does not change in density as the universe expands. Comment on this result.

The fluid equation is

$$\dot{\rho} + 3\frac{\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right) = 0.$$

Constant ρ implies $\dot{\rho} = 0$ so

$$\rho + \frac{p}{c^2} = 0,$$

QED.

9.5. Show that, in the Einstein-de Sitter universe, an object of fixed proper length seen at different redshifts subtends a minimum angle at one particular redshift, and calculate the value of this redshift.

The angle subtended by an object of length l is

$$\alpha = \frac{l}{d_A},$$

where d_A is the angular diameter distance given by

$$d_A = \frac{R_0 S_k(\chi)}{1+z}.$$

The Einstein-de Sitter universe is flat ($k = 0$), so $S_k(\chi) = \chi$. The radial coordinate χ is given by

$$\chi = \int_{t_e}^{t_0} \frac{c\,dt}{R(t}.$$

For an Einstein-de Sitter universe

$$R(t) = R_0 \frac{t^{2/3}}{t_0^{2/3}},$$

therefore

$$\begin{aligned} R_0\chi &= c t_0^{2/3} \int_{t_e}^{t_0} t^{-2/3}\,dt, \\ &= 3 c t_0^{2/3} \left(t_0^{1/3} - t_e^{1/3} \right), \\ &= 3 c t_0 \left(1 - \left(\frac{t_e}{t_0} \right)^{1/3} \right), \\ &= 3 c t_0 \left(1 - (1+z)^{-1/2} \right). \end{aligned}$$

Combining all the results

$$\alpha = \frac{l(1+z)}{3 c t_0 \left(1 - (1+z)^{-1/2}\right)}.$$

For large z, the denominator becomes constant while the numerator grows, and since for small z (small distances) α must drop with z, there must be a minimum. Ignoring the constants and taking the derivative

$$\frac{1}{1-(1+z)^{-1/2}} - \frac{(1+z)(1+z)^{-3/2}}{2\left(1-(1+z)^{-1/2}\right)^2} = 0.$$

Therefore

$$2\left(1 - (1+z)^{-1/2}\right) = (1+z)^{-1/2},$$

or

$$(1+z)^{-1/2} = \frac{2}{3},$$

which gives $z = 5/4 = 1.25$.

Estimate the minimum angular diameter of a galaxy of diameter 20 kpc assuming that our Universe follows the Einstein-de Sitter model with $H_0 = 72\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

For $z = 5/4$,

$$\alpha = \frac{l(1+5/4)}{3ct_0(1-(1+5/4)^{-1/2})} = \frac{9l}{4ct_0}.$$

*From Q**9.2**, $t_0 = 2/3H_0$, so*

$$\alpha = \frac{27lH_0}{8c} = \frac{27 \times 2 \times 10^4 \times 72 \times 10^{-6}}{8 \times 3 \times 10^5} = 3.3".$$

9.6. The different components of the Universe (matter, radiation and the cosmological constant) all have equations of state of the form $p = w\rho c^2$.

(a) Write down the values of w for each component.

$w_M = 0$, $w_R = 1/3$, $w_\Lambda = -1$.

(b) Use the acceleration equation to obtain a condition on w for a component if it is to accelerate the rate of expansion of the Universe. (The density is always assumed to be positive.)

Acceleration equation

$$\ddot{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) R,$$

so for $\ddot{R} > 0$, we must have

$$\rho + \frac{3p}{c^2} < 0,$$

or

$$p < -\frac{1}{3}\rho c^2.$$

Thus $w < -1/3$ is the condition for a component which can accelerate the expansion.

9.7. The "surface brightness" of an object is the flux from it measured at Earth per unit solid angle or equivalently square arcsecond on the sky. Show that the surface brightness of the same object seen at different redshifts scales as $(1+z)^{-4}$.

The flux from an object scales as

$$f = \frac{L}{4\pi d_L^2}.$$

If the object is a square of side l, its solid angle is

$$\Omega = \alpha^2 = \frac{l^2}{d_A^2},$$

so the surface brightness

$$S = \frac{f}{\Omega} = \frac{L}{4\pi l^2}\frac{d_A^2}{d_L^2}.$$

The final factor is

$$\frac{d_A^2}{d_L^2} = \frac{R_0^2 S_k^2(\chi)(1+z)^{-2}}{R_0^2 S_k^2(\chi)(1+z)^2} = (1+z)^{-4},$$

QED.

9.8. Calculate the radius of a sphere of density equal to the critical density that contains a mass equal to that of the Sun, assuming that $H_0 = 72\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

The critical density is given by

$$\rho_c = \frac{3H_0^2}{8\pi G} = 9.7 \times 10^{-27}\,\mathrm{kg\,m^{-3}}.$$

The radius is thus given by

$$R = \left(\frac{3\mathrm{M}_\odot}{4\pi\rho_c}\right)^{1/3} = 3..66 \times 10^{18}\,\mathrm{m} = 118\,\mathrm{pc}.$$

The galaxy is very over-dense compared to the average because in a sphere of this size there are hundreds of thousands of stars.

9.9. (a) Show that for a universe governed by matter and the cosmological constant alone, Friedmann's equation can be written as

$$H(z) = H_0\left(\Omega_M(1+z)^3 + \Omega_\Lambda - (\Omega_M + \Omega_\Lambda - 1)(1+z)^2\right)^{1/2}.$$

where H_0 is the present day value of Hubble's constant, $H(z)$ is Hubble's constant as a function of redshift z and Ω_M and Ω_Λ are the present day ratios of the matter and cosmological constant densities to the critical density $\rho_C = 3H_0^2/8\pi G$.

Friedmann's equation can be written

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}.$$

The density can be written as

$$\rho = \rho_m + \rho_\lambda,$$

where lower case letters represent time variable values as opposed to present day values ρ_M, ρ_Λ. Since $\rho_m \propto R^{-3}$ while ρ_λ is constant, and because $R \propto (1+z)^{-1}$, we can write

$$\rho = \rho_M(1+z)^3 + \rho_\Lambda = \rho_c \left(\Omega_M(1+z)^3 + \Omega_\Lambda \right).$$

Hence, since $\rho_c = 3H_0^2/8\pi G$, Friedmann's equation becomes

$$H^2 = H_0^2 \left(\Omega_M(1+z)^3 + \Omega_\Lambda \right) - \frac{kc^2}{R^2}.$$

Setting $H = H_0$, $R = R_0$ for the present day when $z = 0$, we have

$$\frac{kc^2}{R_0^2} = H_0^2(\Omega_M + \Omega_\Lambda - 1),$$

thus

$$\frac{kc^2}{R^2} = \frac{kc^2}{R_0^2}\left(\frac{R_0}{R}\right)^2 = H_0^2(\Omega_M + \Omega_\Lambda - 1)(1+z)^2.$$

Hence Friedmann's equation becomes

$$H^2 = H_0^2 \left(\Omega_M(1+z)^3 + \Omega_\Lambda - (\Omega_M + \Omega_\Lambda - 1)(1+z)^2 \right).$$

(b) Which term in the above relation represents curvature?

The term in $(1+z)^2$ is the curvature.

(c) When discussing distances, the following integral was needed to evaluate the comoving radial coordinate χ:

$$\chi = \int_0^z \frac{c\,dz}{H(z)}.$$

Show from the relation of part (a) that this integral is a monotonically increasing function of Ω_Λ.

$$\chi = \int_0^z \frac{c\,dt}{H_0 \left(\Omega_M(1+z)^3 + \Omega_\Lambda - (\Omega_M + \Omega_\Lambda - 1)(1+z)^2 \right)^{1/2}}.$$

The Λ-dependent part in the denominator is

$$\Omega_\Lambda(1 - (1+z)^2) < 0,$$

for $z > 0$. Thus as the denominator decreases with Λ, and so the integral increases.

(d) What is the relation of the result of part (c) to the use of supernovae in cosmology?

Distant supernovae appear to dim faster than expected on a matter-only model and are better fitted if there is a significant cosmological constant.

9.10. (a) Starting from Einstein's field equations in the contravariant form

$$R^{\alpha\beta} = -k\left(T^{\alpha\beta} - \frac{1}{2}Tg^{\alpha\beta}\right) + \Lambda g^{\alpha\beta},$$

and given that for the FRW metric

$$R^{tt} = \frac{3}{c^4}\frac{\ddot{R}}{R},$$

derive the following form of the acceleration equation

$$\ddot{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)R + \frac{1}{3}\Lambda c^2 R.$$

where ρ does *not* include a cosmological constant component.

Assuming a perfect fluid

$$T^{\alpha\beta} = \left(\rho + \frac{p}{c^2}\right)U^\alpha U^\beta - pg^{\alpha\beta}.$$

In co-moving coordinates the fluid is stationary so $U^i = 0$ *and*

$$g_{\alpha\beta}U^\alpha U^\beta = g_{tt}U^t U^t = c^2.$$

In the FRW metric, coordinates (t, r, θ, ϕ), $g_{tt} = c^2$, *so* $U^t = 1$ *and* $g^{tt} = 1/c^2$, *so*

$$T^{tt} = \rho + \frac{p}{c^2} - \frac{p}{c^2} = \rho.$$

The trace T *is defined by*

$$T = g_{\alpha\beta}T^{\alpha\beta} = \left(\rho + \frac{p}{c^2}\right)g_{\alpha\beta}U^\alpha U^\beta - pg_{\alpha\beta}g^{\alpha\beta} = \rho c^2 + p - 4p = \rho c^2 - 3p.$$

Therefore the field equations for the tt component can be written

$$\frac{3}{c^4}\frac{\ddot{R}}{R} = -k\left(\rho - \frac{1}{2}(\rho c^2 - 3p)c^{-2}\right) + \Lambda c^{-2},$$

and thus

$$\frac{\ddot{R}}{R} = -\frac{kc^4}{6}\left(\rho + \frac{3p}{c^2}\right) + +\frac{1}{3}\Lambda c^2.$$

Finally putting $k = 8\pi G/c^4$, *and multiplying through by (a) we get*

$$\ddot{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)R + +\frac{1}{3}\Lambda c^2 R.$$

(b) Show that this is equivalent to the equation derived in lectures if one adopts the view that the cosmological constant is a fluid of density

$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi G},$$

and pressure $p_\Lambda = -\rho_\Lambda c^2$.

Starting from

$$\ddot{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) R,$$

set $\rho \to \rho + \rho_\Lambda$ and $p \to p + p_\Lambda$, and substitute for ρ_Λ and p_Λ using the expressions given, lead to

$$\begin{aligned}\ddot{R} &= -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) R - \frac{4\pi G}{3}\left(\frac{\Lambda c^2}{8\pi G} - 3\frac{\Lambda c^2}{8\pi G}\right) R, \\ &= -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) R + \frac{1}{3}\Lambda c^2 R.\end{aligned}$$

I find the acceleration equation without Λ easier to remember, along with the relation $p_\Lambda = -\rho_\Lambda c^2$, although I tend not to remember the expression for ρ_Λ in terms of Λ.

9.11. There are no temperature gradients in a homogeneous Universe and so no heat transfer and thus one can write

$$dU + p\, dV = 0,$$

where U is the internal energy of a volume V of the Universe and p is the pressure. Use this and the relativistic mass-energy relation to prove the fluid equation:

$$\dot{\rho} + 3H(t)\left(\rho + \frac{p}{c^2}\right) = 0,$$

where ρ is the density, p is the pressure, $H(t)$ is Hubble's "constant" as a function of time and the dot denotes a derivative with respect to cosmic time t.

The volume of any region of the Universe scales as $V \propto R^3$, and therefore setting $V = kR^3$ where k is a constant (not the curvature constant) then

$$\frac{dV}{V} = 3\frac{dR}{R}.$$

From $E = mc^2$, $U = V\rho c^2$,

$$dU = \rho c^2\, dV + V c^2\, d\rho.$$

Hence dividing through by V,

$$\rho c^2 \frac{dV}{V} + c^2\, d\rho + p\frac{dV}{V} = 0,$$

and hence dividing through by c^2

$$d\rho + \left(\rho + \frac{p}{c^2}\right)\frac{dV}{V} = 0.$$

Finally using the relation for dV/V*, dividing through by* dt *and recognising* $H = \dot{R}/R$ *the fluid equation as given results.*

9.12. "CODEX" is a proposed high-resolution spectrograph designed to measure the rate of change of the redshift $\dot{z}$ of distant objects as a way to measure past values of Hubble's constant. This is in principle visible as a change in the apparent recession velocity of the objects.

(a) Show that

$$\dot{z} = (1+z)H_0 - H(z),$$

where H_0 is the present day value of Hubble's constant and $H(z)$ is the value of Hubble's constant at the time light from objects at redshift z was emitted.

The fundamental relation for redshift is

$$1 + z = \frac{R(t_o)}{R(t_e)},$$

where t_o *is the present age of the Universe* $= t$, t_e *is the age when the light from the quasar was emitted. Thus*

$$\frac{dt_o}{dt} = 1,$$

and

$$\frac{dt_e}{dt} = \frac{1}{1+z},$$

the second relation following from consideration of the same proof that leads to the redshift relation.
Therefore

$$\begin{aligned}
\dot{z} &= \frac{\dot{R}(t_o)}{R(t_e)}\frac{dt_o}{dt} - \frac{R(t_o)\dot{R}(t_e)}{R^2(t_e)}\frac{dt_e}{dt}, \\
&= \frac{R(t_o)}{R(t_e)}\left(\frac{\dot{R}(t_o)}{R(t_o)} - \frac{\dot{R}(t_e)}{R(t_e)}\frac{1}{1+z}\right), \\
&= (1+z)H_0 - H(z).
\end{aligned}$$

where $H_0 = \dot{R}(t_0)/R(t_0)$ *and* $H(z) = \dot{R}(t_e)/R(t_e)$*. QED.*

(b) Hence calculate the rate of change of recession velocity of an object of redshift $z = 1$, assuming an Einstein-de Sitter universe with $R(t) \propto t^{2/3}$.
[Present day value of Hubble's constant $H_0 = 72\,\mathrm{km\,s^{-1}Mpc^{-1}}$.]

For the Einstein-de Sitter model

$$H(z) = \frac{\dot{R}(t_e)}{R(t_e)} = \frac{2}{3t_e}.$$

Hence

$$\begin{aligned} H(z) &= \frac{2}{3t_e}, \\ &= \frac{2}{3t_0}\frac{t_0}{t_e}, \\ &= H_0(1+z)^{3/2}, \end{aligned}$$

since $1+z = R(t_0)/R(t_e) = (t_0/t_e)^{2/3}$. *The apparent change in velocity is given by*

$$\dot{v} = \frac{c\dot{\lambda_0}}{\lambda_0} = \frac{c\lambda_e \dot{z}}{\lambda_0} = \frac{c\dot{z}}{1+z}.$$

This gives $\dot{v} = -0.9\,\mathrm{cm\,s^{-1}\,yr^{-1}}$, *a very challenging number!*

9.13. A ballistic projectile is fired from the origin in an Einstein-de Sitter universe ($R(t) \propto t^{2/3}$) towards a galaxy nearby enough to have a non-relativistic recession speed due to universal expansion.

(a) Show that

$$\ddot{\chi} = -2\frac{\dot{R}}{R}\dot{\chi},$$

where the dots denote derivatives with respect to the proper time τ of the projectile.

Ignoring angular terms, the Lagrangian is given by

$$L = c^2\dot{t}^2 - R^2\dot{\chi}^2.$$

The Euler-Lagrange equations then lead to the radial equation of motion:

$$\frac{d}{d\tau}\left(-2R^2\dot{\chi}\right) = 0,$$

so

$$R^2\ddot{\chi} = -2R\dot{R}\dot{\chi},$$

or

$$\ddot{\chi} = -2\frac{\dot{R}}{R}\dot{\chi},$$

where the derivatives are with respect to the proper time of the projectile.

(b) * Hence show that in order for the projectile to reach the galaxy it must be fired at a speed greater than half the apparent recession speed of the galaxy as measured at the time of firing.

From the Lagrangian

$$c^2\dot{t}^2 - R^2\dot{\chi}^2 = c^2.$$

Now

$$\frac{d\chi}{d\tau} = \frac{dt}{d\tau}\frac{d\chi}{dt} = \dot{t}\frac{d\chi}{dt},$$

so

$$\dot{t}^2\left(1 - \frac{R^2\dot{\chi}^2}{c^2}\right) = 1.$$

The projectile's proper distance $d = R\chi$ increases at a rate

$$\begin{aligned}
\frac{d(R\chi)}{dt} &= \dot{R}\chi + R\dot{\chi}, \\
&= \frac{\dot{R}}{R}R\chi + R\dot{\chi}, \\
&= Hd + R\dot{\chi},
\end{aligned}$$

thus $v = R\dot{\chi}$ is the amount by which the projectile's proper distance increases in excess of the recession speed at the point the projectile has reached. Thus

$$\dot{t} = \frac{dt}{d\tau} = \frac{1}{\sqrt{1 - v^2/c^2}},$$

a familiar relation. Since the galaxy's recession speed is $\ll c$, we can therefore assume that $\dot{t} = 1$, or $\tau = t$. Therefore with $R \propto t^{2/3}$, the equation of the previous part can be written

$$\frac{d\dot{\chi}}{dt} = -\frac{4}{3t}\dot{\chi}.$$

This leads to

$$\ln\dot{\chi} = -\frac{4}{3}\ln t + k,$$

where k is a constant, so

$$\dot{\chi} = \frac{v_P}{R_0}\left(\frac{t}{t_0}\right)^{-4/3}.$$

Integrating again

$$\begin{aligned}
\chi &= \frac{v_P t_0^{4/3}}{R_0}\left[-3t^{-1/3}\right]_{t_0}^{\infty}, \\
&= \frac{3v_P t_0}{R_0}, \\
&= \frac{2v_P}{H_0 R_0}.
\end{aligned}$$

Thus the projectile reaches a proper distance (as measured at the time it is fired) $d = 2v_P/H_0$. At the start, a point at this distance recedes from the origin at rate $H_0 d = 2v_P$. Thus as long as $v_P > v_G/2$, the projectile will catch up with the galaxy.

10.1. The generalised coordinates of GR can be confusing; the TT gauge of gravitational waves is a good example of this. This question illustrates this. A weak gravitational field is described by $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ where $|h_{\alpha\beta}| \ll 1$.

(a) Use the Levi-Civita equation (handout 3) to show that to first order the connection can be written

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2}\eta^{\alpha\sigma}\left(h_{\beta\sigma,\gamma} + h_{\sigma\gamma,\beta} - h_{\beta\gamma,\sigma}\right).$$

This is straightforward. The Levi-Civita equation is

$$\Gamma^{\alpha}{}_{\beta\gamma} = \frac{1}{2}g^{\alpha\sigma}\left(g_{\sigma\gamma,\beta} + g_{\beta\sigma,\gamma} - g_{\beta\gamma,\sigma}\right),$$

and retaining only first order terms the result follows immediately (derivatives of the SR metric η are zero).

(b) In the TT gauge, $h_{\alpha 0} = h_{0\alpha} = 0$. Use the general equations of motion

$$\ddot{x}^{\alpha} + \Gamma^{\alpha}{}_{\beta\gamma}\dot{x}^{\beta}\dot{x}^{\gamma} = 0,$$

and the result of part (a) to show that freely-floating particles which are initially stationary ($\dot{x}^i = 0$, $i = 1, 2, 3$) will remain stationary in the presence of gravitational waves.

With $\dot{x}^i = 0$, the equations of motion become

$$\ddot{x}^{\alpha} = -\Gamma^{\alpha}{}_{00}\dot{x}^0\dot{x}^0.$$

However from part (a),

$$\Gamma^{\alpha}{}_{00} = \frac{1}{2}\eta^{\alpha\sigma}\left(h_{0\sigma,0} + h_{\sigma 0,0} - h_{00,\sigma}\right) = 0,$$

because of the TT gauge conditions. Therefore

$$\ddot{x}^{\alpha} = 0,$$

and so the particles remain stationary in the TT coordinates.

(c) The previous result is misleading: the particles are stationary in *coordinates*, but the physically measurable distance between particles is variable, i.e. the TT coordinates track the particles. To see this show, from the form of $h^{\alpha\beta}$ derived in lectures that the distance l from the origin of a free particle at TT coordinates (x, y) as a gravitational wave of angular frequency Ω passes in the z direction is given by $l^2 = x^i A_{ij} x^j$, where i and j are only summed over x and y and the 2x2 matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 1 - a\cos\Omega t & -b\cos\Omega t \\ -b\cos\Omega t & 1 + a\cos\Omega t \end{pmatrix}.$$

(In general the time-dependent terms could also include phase shifts.)

Setting $dt = 0$, and $l^2 = -s^2$, the distance measured from the origin is given by

$$l^2 = -g_{ij} x^i x^j.$$

where i and j only apply to x and y. Since $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, and remembering that $\eta_{xx} = \eta_{yy} = -1$, then

$$A_{ij} = -g_{ij} = \delta_{ij} - h_{ij}.$$

Given the form of h_{ij} from lectures (an remembering that it oscillates), the results follow.

(d) Use the relation of part (c) to justify the standard elliptical distortion patterns of gravitational waves when they pass through a set of particles initially arranged in a circle.

$l^2 = x^i A_{ij} x^j$ is what is known as a quadratic form, and as is well-known from matric theory can be expressed as $l^2 = \sum_{i=1}^{2} \lambda_i u_i^2$ where λ_i are the eigenvalues of $\mathbf{A}$ and u_i are the components of (x, y) resolved along the corresponding eigenvectors. The eigenvectors are perpendicular to each other since $\mathbf{A}$ is symmetric. If $\lambda_1 \neq \lambda_2$, a set of points forming a circle are evidently deformed into an ellipse. Consider first the case when $b = 0$, so that

$$\mathbf{A} = \begin{pmatrix} 1 - a\cos\Omega t & 0 \\ 0 & 1 + a\cos\Omega t \end{pmatrix}.$$

Then the eigenvectors are clearly $(1, 0)$ and $(0, 1)$, so this corresponds to the polarisation where the stretching and squeezing occur along the x and y axes. For $a = 0$ we have

$$\mathbf{A} = \begin{pmatrix} 1 & b\cos\Omega t \\ b\cos\Omega t & 1 \end{pmatrix}.$$

which has eigenvectors $(1, 1)$ and $(1, -1)$ so that the ellipses are oriented at $45°$ degrees to the x and y axes. These two polarisations are independent and need not be in phase with each other.

10.2. Using the formula from lectures for the amplitude of gravitational waves produced by a variable mass quadrupole at distance r

$$\bar{h}^{ij} = -\frac{2G}{c^4 r}\frac{d^2 I^{ij}}{dt^2},$$

where

$$I^{ij} = \int \rho x^i x^j \, dV,$$

discuss the feasibility of setting up a calibration source for the current laser interferometers consisting of two equal masses rotated around a vertical axis. Assume a sensitivity of $h \sim 10^{-21}$ at frequencies $\sim 100\,\text{Hz}$.

Assuming that each mass m is a distance a from the axis then their x coordinates are given by

$$x = \pm a \cos \omega t.$$

Assuming point masses, the quadrupole integral reduces to a sum over each mass:

$$I^{xx} = m(a \cos \omega t)^2 + m(a \cos \omega t)^2 = ma^2(1 + \cos 2\omega t).$$

Differentiating twice,

$$\ddot{I}^{xx} = -4ma^2\omega^2 \cos 2\omega t.$$

Therefore the $\bar{h}^{xx}$ component oscillates with amplitude

$$\frac{8Gma^2\omega^2}{c^4 r}.$$

For a given target h this defines the parameters required for the calibration source. The interferometers are several kilometres in length so let's take $r = 100\,\text{km}$ to make sure the waves produced by the calibration source are roughly planar. Therefore

$$ma^2\omega^2 = \frac{c^4 hr}{8G} \sim 10^{26}\,\text{J}.$$

This corresponds to $\sim 10^{16}\,\text{J}$ per person on the planet, equivalent to $300\,MW$ per person for a year. This is not going to happen!

NB Strictly speaking, one should impose the condition that the solution is traceless by correcting by $-(I^{xx} + I^{yy})/2$, but this makes no qualitative difference to the final conclusion.

10.3. Two particles of equal mass M separated by a moving in circular orbits around their centre of mass lose energy due to the emission of gravitational waves at a rate

$$\frac{dE}{dt} = -\frac{64G^4M^5}{5c^5a^5}.$$

The energy comes from the shrinkage of the orbit of the two particles.

(a) Using Newtonian mechanics show that the rate of change of the orbital separation is given by

$$\dot{a} = -\frac{128G^3M^3}{5c^5a^3}.$$

By the virial theorem or by summing potential and kinetic energies, the total energy of the system is

$$E = -\frac{GM^2}{2a}.$$

Taking its derivative

$$\frac{dE}{dt} = \frac{GM^2}{2a^2}\dot{a}.$$

The formula for $\dot{a}$ then follows directly.

(b) Hence obtain an expression for the time taken for the two particles to spiral together.

Integrating

$$\int_a^0 a^3\,da = -\frac{128G^3M^3}{5c^5}\int_0^t dt,$$

and thus

$$t = \frac{5c^5a^4}{512G^3M^3}.$$

(c) Calculate the time for two neutron stars, each with $M = 1.4\,\mathrm{M}_\odot$ to merge starting from an orbital period of 2 hr. [Assume circular orbits throughout.]

Kepler 3 gives

$$\Omega^2 = \frac{4\pi}{P^2} = \frac{G(2M)}{a^3},$$

which gives $a = 7.886 \times 10^8\,\mathrm{m}$. *This then gives* $t = 4.4 \times 10^7\,\mathrm{yr}$, *relatively short-lived by astronomical standards, so gravitational waves act to remove short period binaries, or force them to evolve.*

(d) The LIGO interferometers are expected to pick up the final in-spiral of pairs of neutron stars when they have reached a gravitational wave frequency of about 50 Hz; seismic noise makes detection difficult at lower frequencies.

Estimate how long a neutron star merger event will last for LIGO.

The time to merger scales as a^4 while from Kepler's third law, $a \propto P^{2/3}$, thus $t \propto P^{8/3}$. 50 Hz in gravitational waves corresponds to an orbital frequency of 25 Hz, so $P = 0.04\,\mathrm{s}$. Thus the merger time compared to the previous part is

$$(4.4 \times 10^7\,\mathrm{yr}) \times \left(\frac{0.04}{7200}\right)^{8/3} = 13.4\,\mathrm{s}.$$

Would a pair of merging $10\,\mathrm{M}_\odot$ black-holes be detectable over a longer or shorter interval of time?

The merger time scales as

$$t \propto \frac{a^4}{M^3},$$

while $a \propto M^{1/3}P^{2/3}$, thus

$$t \propto M^{-5/3}P^{8/3}.$$

Assuming that the black-holes are also first detected at 50 Hz, then they will be seen for a shorter interval of time. In reality the frequency at which they are seen will depend upon their distance, but since the seismic noise rises steeply, this is probably a reasonable approximation.

(e) Qualitatively describe the nature of the gravitational wave signal that will characterise the first part of such mergers, while the two stars can be treated as point masses.

The signal's frequency will get higher and higher with time, and since

$$h \propto a^2\Omega^2,$$

while $\Omega^2 \propto a^{-3}$, then $h \propto a^{-1}$, and the strength of the signal will increase with time too as the separation decreases. This is known as the "chirp" signal.

10.4. Use the quadrupole formula to show that a spherically symmetric mass distribution produces no gravitational waves.

Without loss of generality we can assume that we are located on the z-axis, and we can take the origin to be the centre of symmetry. We are interested in integrals of the form $I^{xx} = \int \rho x^2\, dV$, $I^{yy} = \int \rho y^2\, dV$ and $I^{xy} = I^{yx} = \int \rho xy\, dV$. The last integral is clearly zero by spherical symmetry, while the other two must be equal, giving a wave tensor of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

but in the TT gauge, this must be traceless, so $2a = 0$, and so there is no wave produced. Sadly, this means that rather little of the tremendous energy of supernova explosions need go into gravitational waves.

10.5. In lectures (see also problem sheet 5) it was stated that the linearised field equations reduce to

$$h_{,\alpha\beta} + \Box h_{\alpha\beta} - \eta^{\gamma\delta}\left(h_{\alpha\gamma,\delta\beta} + h_{\delta\beta,\alpha\gamma}\right) - \left(\Box h - h^{\sigma\rho}{}_{,\sigma\rho}\right)\eta_{\alpha\beta} = 2kT_{\alpha\beta},$$

where $\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta$.

Show by applying the Lorenz gauge condition:

$$h^{\alpha\beta}{}_{,\beta} = \frac{1}{2}\eta^{\alpha\beta}h_{,\beta},$$

that the field equations reduce to

$$\Box h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\Box h = 2kT_{\alpha\beta}.$$

The first part of the third term in the linearised FEs can be written as

$$\eta^{\gamma\delta}h_{\alpha\gamma,\delta\beta} = \eta_{\alpha\gamma}h^{\gamma\delta}{}_{,\delta\beta} = \frac{1}{2}\eta_{\alpha\gamma}\eta^{\gamma\delta}h_{,\delta\beta} = \frac{1}{2}h_{,\alpha\beta},$$

where the third term makes use of the gauge condition. Similarly:

$$\eta^{\gamma\delta}h_{\delta\beta,\alpha\gamma} = \frac{1}{2}h_{,\beta\alpha}.$$

Given the commutativity of partial derivatives, the first and third term of the FEs cancel leaving:

$$\Box h_{\alpha\beta} - \left(\Box h - h^{\sigma\rho}{}_{,\sigma\rho}\right)\eta_{\alpha\beta} = 2kT_{\alpha\beta}.$$

Again using the gauge condition, the second term in brackets can be written

$$h^{\sigma\rho}{}_{,\sigma\rho} = \frac{1}{2}\eta^{\sigma\rho}h_{,\sigma\rho} = \frac{1}{2}\eta^{\sigma\rho}\partial_\sigma\partial_\rho h = \frac{1}{2}\Box h.$$

The final result follows directly.

10.6. This question concerns the equivalent in GR of magnetic fields, something that does not come up in Newtonian gravity.

(a) Consider a stress-energy tensor $T^{\alpha\beta}$ that is time-independent ($T^{\alpha\beta}{}_{,0} = 0$, often called "stationary" although there can still be motion, e.g. rotation of a sphere).
In this case, justify why the general solution of the linearised field equations

$$\Box\bar{h}^{\alpha\beta} = 2kT^{\alpha\beta},$$

can be written as

$$\bar{h}^{\alpha\beta}(\mathbf{x}) = \frac{k}{2\pi}\int \frac{T^{\alpha\beta}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,d^3y,$$

where $\mathbf{x}$ is the 3-vector position at which we want the value of $\bar{h}^{\alpha\beta}$, and $\mathbf{y}$ defines the volume element while $k = -8\pi G/c^4$.

Since the stress-energy tensor is invariant, the wave operator $\Box$ reduces simply to $-\nabla^2$, and the linearised field equations are

$$\nabla^2\bar{h}^{\alpha\beta} = -2kT^{\alpha\beta},$$

directly analogous to

$$\nabla^2\phi = 4\pi G\rho,$$

which has solution

$$\phi(\mathbf{x}) = -G \int \frac{\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\, d^3y.$$

Adjusting the constant outside the integral, and replacing ρ by $T^{\alpha\beta}$ and ϕ by $\bar{h}^{\alpha\beta}$ leads to the solution given.

(b) For non-relativistic sources, the energy-momentum tensor components are easily seen to be given by

$$T^{00} = \rho c^2, \quad T^{0i} = \rho c u^i, \quad T^{ij} = \rho u^i u^j,$$

where u^i are the components of the 3-velocity. Show from these that to first order in the velocities,

$$\bar{h}^{00} = \frac{4\phi}{c^2}, \quad \bar{h}^{0i} = \frac{A^i}{c}, \quad \bar{h}^{ij} = 0,$$

where

$$\begin{aligned} \phi(\mathbf{x}) &= -G \int \frac{\rho(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\, d^3y, \\ A^i(\mathbf{x}) &= -\frac{4G}{c^2} \int \frac{\rho(\mathbf{y})u^i(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\, d^3y. \end{aligned}$$

Putting $k = -8\pi G/c^4$ then

$$\bar{h}^{\alpha\beta}(\mathbf{x}) = -\frac{4G}{c^4} \int \frac{T^{\alpha\beta}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\, d^3y.$$

This immediately leads to the expressions given, with the component T^{ij} zero to first order in the velocities.

(c) Hence show that

$$h_{00} = h_{11} = h_{22} = h_{33} = \frac{2\phi}{c^2}, \quad h_{0i} = \frac{A_i}{c}.$$

The "trace reverse" goes both ways, i.e.

$$h^{\alpha\beta} = \bar{h}^{\alpha\beta} - \frac{1}{2}\bar{h}\eta^{\alpha\beta},$$

with

$$\bar{h} = h^{\alpha\beta}\eta_{\alpha\beta}.$$

Given the components of $\bar{h}^{\alpha\beta}$, $\bar{h} = \frac{4\phi}{c^2}$, so

$$h^{00} = \frac{4\phi}{c^2} - \frac{1}{2}\frac{4\phi}{c^2}\eta^{00} = \frac{2\phi}{c^2},$$

since $\eta^{00} = 1$. Also since $\eta^{00}v = 1$ and η is diagonal, $h_{00} = h^{00} = 2\phi/c^2$. The other components follow in the same manner.

(d) Finally, show that the corresponding approximate line element can be written as

$$ds^2 = c^2 \left(1 + \frac{2\phi}{c^2}\right) dt^2 + 2A_i \, dt \, dx^i - \left(1 - \frac{2\phi}{c^2}\right) dx^i dx^i,$$

with implied summation over the i-index in the final term.

This follows directly from $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ and the values of $h_{\alpha\beta}$ from the previous part.

The term in A_i shows that in GR, *motion* of the gravitating mass can affect the line element and hence the orbital motion of nearby test particles. This crops up in the *Kerr metric* for rotating black-holes which unfortunately we do not have the time to cover. ϕ and A_i are analogous to the scalar and vector potentials of electromagnetism.