**7.1**. A particle is dropped from rest at a radius $r = r_0$ in Schwarzschild coordinates from a black-hole of mass $M$.

(a) Show that at radius $r$

$$\frac{dr}{d\tau} = -\sqrt{\frac{2GM}{r} - \frac{2GM}{r_0}},$$

where $\tau$ is the particle's proper time.

---

*From the lectures, the "energy equation" for the Schwarzschild metric is*

$$\dot{r}^2 + \frac{h^2}{r^2}\left(1 - \frac{2GM}{c^2 r}\right) - \frac{2GM}{r} = c^2(k^2 - 1).$$

*Where* $h = r^2\dot{\phi}$ *is a constant as is* $k$*. If dropped from rest,* $h = 0$*, and since* $\dot{r} = 0$ *at* $r = r_0$*, we have*

$$-\frac{2GM}{r_0} = c^2(k^2 - 1).$$

*Therefore the energy equation reduces to*

$$\dot{r}^2 - \frac{2GM}{r} = -\frac{2GM}{r_0},$$

*or*

$$\dot{r}^2 = \frac{2GM}{r} - \frac{2GM}{r_0},$$

*and the result follows, remembering that the particle clearly moves towards smaller* $r$*.*

---

(b) Hence show that the total proper time for the particle to reach the singularity at $r = 0$ is given by

$$\tau = \pi\sqrt{\frac{r_0^3}{8GM}}.$$

---

*The proper time is given by*

$$\tau = \int_{r_0}^{0} -\frac{dr}{\sqrt{2GM/r - 2GM/r_0}} = \int_0^{r_0} \frac{dr}{\sqrt{2GM/r - 2GM/r_0}}.$$

*Thus*

$$\tau = \frac{1}{\sqrt{2GM}} \int_0^{r_0} \sqrt{\frac{r_0 r}{r_0 - r}}\, dr.$$

*Making the substitution* $r = r_0 \sin^2\theta$*, the integral becomes*

$$\begin{aligned}\int_0^{r_0} \sqrt{\frac{r_0 r}{r_0 - r}}\, dr &= r_0^{3/2} \int_0^{\pi/2} 2\sin^2\theta\, d\theta = r_0^{3/2}\int_0^{\pi/2} (1 - \cos 2\theta)\, d\theta, \\ &= r_0^{3/2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{\pi}{2} r_0^{3/2}.\end{aligned}$$

*Hence*

$$\tau = \pi\sqrt{\frac{r_0^3}{8GM}}.$$

---

(c) Calculate the proper time taken for a particle dropped from the event horizon of the black-hole at the centre of our Galaxy ($M = 4.5 \times 10^6\,\mathrm{M}_\odot$) to reach the singularity.

---

*Put* $r_0 = R_S = 2GM/c^2$, *then*

$$\tau = \pi\sqrt{\frac{8(GM)^3}{8GMc^6}} = \frac{\pi GM}{c^3} = \frac{\pi \times 6.67 \times 10^{-11} \times 4.5 \times 10^6 \times 2 \times 10^{30}}{(3 \times 10^8)^3} = 69.9\ \mathrm{sec}\,.$$

---

**7.2**. An observer stationary at radius $r > R_S$ (Schwarzschild coordinates) measures the speed of the particle of Q**7.1** as it passes by having started at $r_0 > r$. Show the following:

(a) The proper "ruler" distance $dl$ measured by the observer corresponding to a change in radial coordinate $dr$ is

$$dl = \frac{dr}{(1 - 2\mu/r)^{1/2}},$$

where $\mu = GM/c^2$.

---

*Suppressing the angular coordinates, the Schwarzschild metric is*

$$ds^2 = c^2\left(1 - \frac{2\mu}{r}\right)dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2.$$

*Proper length is given by* $dl^2 = -ds^2$ *for* $dt = 0$, *thus*

$$dl^2 = \frac{dr^2}{1 - 2\mu/r},$$

*and the result follows.*

---

(b) The derivative of coordinate time $t$ with respect to the particle's proper time $\tau_p$ is given by

$$\left(\frac{dt}{d\tau_p}\right)^2 = \dot{t}^2 = \left(1 - \frac{2\mu}{r}\right)^{-1} + \left(1 - \frac{2\mu}{r}\right)^{-2}\left(\frac{\dot{r}}{c}\right)^2.$$

---

*The 4-velocity norm implies that*

$$g_{tt}\dot{t}^2 + g_{rr}\dot{r}^2 = c^2,$$

*with* $g_{tt} = c^2(1 - 2\mu/r)$ *and* $g_{rr} = -(1 - 2\mu/r)^{-1}$. *Therefore*

$$c^2(1 - 2\mu/r)\dot{t}^2 - (1 - 2\mu/r)^{-1}\dot{r}^2 = c^2.$$

*Given that* $\dot{t} = dt/d\tau_p$, *the result again follows.*

---

(c) The derivative of the observer's proper time $\tau_o$ with respect to coordinate time $t$ is given by

$$\frac{d\tau_o}{dt} = \left(1 - \frac{2\mu}{r}\right)^{1/2}.$$

---

*This follows directly from $c^2\, d\tau_o^2 = ds^2$ and $dr = 0$ for the observer.*

---

(d) Thus, combining the above results, show that if the observer is very close to the event horizon, the particle will pass by at the speed of light, independent of its initial radius.

---

*The observer measures a speed*

$$\begin{aligned} v &= \frac{dl}{d\tau_o} = \frac{dl}{dr}\frac{dr}{d\tau_p}\frac{d\tau_p}{dt}\frac{dt}{d\tau_o}, \\ &= \frac{(dl/dr)(dr/d\tau_p)}{(dt/d\tau_p)(d\tau_o/dt)}, \\ &= \frac{(1-2\mu/r)^{-1/2}\dot{r}}{(dt/d\tau_p)(1-2\mu/r)^{1/2}}, \\ &= \frac{(1-2\mu/r)^{-1}\dot{r}}{(dt/d\tau_p)}, \end{aligned}$$

*The equation for $dt/d\tau_p$ is dominated by the second term as $r \to 2\mu$ at the event horizon, and so*

$$\frac{dt}{d\tau_p} \to \left(1 - \frac{2\mu}{r}\right)^{-1}\frac{\dot{r}}{c},$$

*and hence $v \to c$.*

---

**7.3**. A particle is set in an orbit around a black-hole of mass $M$ starting from radius $r_0 \gg \mu = GM/c^2$ with a purely angular motion at speed $v = r\dot{\phi} = h/r$. The speed $v$ is much less than the circular orbital speed at $r_0$ so that as it orbits the particle passes close to the black-hole.

(a) Using the energy equation from lectures, show that $|k^2 - 1| \ll 1$.

---

*From*

$$\dot{r}^2 + \frac{h^2}{r^2}\left(1 - \frac{2\mu}{r}\right) - \frac{2\mu c^2}{r} = c^2(k^2 - 1),$$

*setting $\dot{r} = 0$ and $h = r_0 v$, and using $r_0 \gg \mu$, we have*

$$c^2(k^2 - 1) \approx v^2 - \frac{2\mu c^2}{r_0}.$$

*The circular orbital speed far from the black-hole is given by the Newtonian equation*

$$v_c^2 = \frac{GM}{r_0} = \frac{\mu c^2}{r_0}.$$

*Since $v \ll v_C$, we can write*

$$c^2(k^2 - 1) \approx -\frac{2\mu c^2}{r_0},$$

*so*

$$|k^2 - 1| \approx \frac{2\mu}{r_0} \ll 1.$$

---

(b) Hence, by considering the effective potential, show that the particle will be captured by the black-hole if

$$v < 4\left(\frac{\mu}{r_0}\right)c.$$

---

*For usual values of $h$, as one proceeds from large to small radii, the effective potential*

$$V_{eff} = \frac{h^2}{2r^2}\left(1 - \frac{2\mu}{r}\right) - \frac{\mu c^2}{r},$$

*goes through a minimum and then a maximum, before finally plunging down towards $r = 0$ as the $1/r^3$ term dominates. Therefore as long as the total energy is enough to exceed the inner maximum, the particle will be captured. From the first part, this is the case if the inner maximum is less than $\approx 0$, since the total energy of the particle is very small. $V_{eff} = 0$ implies that*

$$\frac{h^2}{2r^2}\left(1 - \frac{2\mu}{r}\right) - \frac{\mu c^2}{r} = 0,$$

*and so multiplying by $-r^3$ gives the quadratic*

$$2\mu c^2 r^2 - h^2 r + 2\mu h^2 = 0.$$

*This will fail to reach 0 if "$b^2 < 4ac$" or*

$$h^4 < 16\mu^2 c^2 h^2,$$

*or*

$$h < 4\mu c.$$

*The result follows immediately.*

---

(c) Calculate the maximum value of $v$ for $M = 1\,\mathrm{M}_\odot$, and $r_0 = 1\,\mathrm{AU}$.

---

*For $M = 1\,\mathrm{M}_\odot$, $\mu \approx 1.5\,\mathrm{km}$, while $1\,\mathrm{AU} \approx 1.5 \times 10^8\,\mathrm{km}$, so $v < 4 \times 10^{-8}c = 12\,\mathrm{m\,s^{-1}}$. If one compares with Earth's orbital speed of $\approx 30\,\mathrm{km\,s^{-1}}$, it can be seen that it is not that easy to give an object a low enough angular momentum to fall into a black-hole, which is why accretion discs are astrophysically important.*

---

**7.4**. Show that the radius of the inner maximum of the Schwarzschild potential – provided that there is such a maximum – is minimised as $h \to \infty$.

---

*The Schwarzschild potential is given by*

$$V(r) = \frac{h^2}{2r^2}\left(1 - \frac{2\mu}{r}\right) - \frac{\mu c^2}{r}.$$

*Extrema are defined by* $V'(r) = 0$ *or*

$$V'(r) = -\frac{h^2}{r^3} + \frac{3\mu h^2}{r^4} + \frac{\mu c^2}{r^2} = 0,$$

*which gives*

$$\mu c^2 r^2 - h^2 r + 3\mu h^2 = 0,$$

*so the maximum is at radius*

$$r = \frac{1}{2\mu c^2}\left[h^2 - \left(h^4 - 12\mu^2 c^2 h^2\right)^{1/2}\right].$$

*Setting* $x = h^2$ *and taking the derivative* $2\mu c^2 dr/dx$ *gives*

$$\begin{aligned} 2\mu c^2 \frac{dr}{dx} &= 1 - \frac{x - 6\mu c^2}{\left(x^2 - 12\mu c^2 x\right)^{1/2}}, \\ &= 1 - \frac{x - 6\mu c^2}{\left((x - 6\mu c^2)^2 - 36\mu^2 c^4\right)^{1/2}}. \end{aligned}$$

*In the final form, so long as* $x = h^2 > 12\mu c^2$ *to ensure the existence of the inner maximum, the numerator in the fraction is positive while the denominator is positive and manifestly smaller than the numerator. Therefore*

$$2\mu c^2 \frac{dr}{dx} < 0.$$

*Hence* $r$ *is minimised as* $h \to \infty$.

---

Hence show that the closest one can pass by a black-hole of mass $M$ without being captured is given by $r = 3GM/c^2$.

---

*The minimum radius that the inner maximum reaches represents the closest one can reach without being captured because any particle that manages to get over the inner maximum is doomed. From the expression for* $r$

$$r = \frac{h^2}{2\mu c^2}\left[1 - \left(1 - \frac{12\mu^2 c^2}{h^2}\right)^{1/2}\right].$$

*Letting* $h \to \infty$, *and expanding the last term*

$$r \to \frac{h^2}{2\mu c^2}\left[1 - 1 + \frac{6\mu^2 c^2}{h^2}\right] = 3\mu = \frac{3GM}{c^2},$$

*QED.*

---

**7.5**. If the Sun was replaced by a black-hole of the same mass, how accurately would a laser from Earth have to be directed to ensure that its beam was captured? [Ignore Earth's motion.]

---

*The energy equation for photons is*

$$\dot{r}^2 + \frac{h^2}{r^2}\left(1 - \frac{2\mu}{r}\right) = c^2 k^2.$$

*The effective potential has a single maximum which is easily shown to be at* $r = 3\mu$. *Assuming that the laser beam is directed at an angle* $\alpha$ *from the direct line to the black-hole, we can say that*

$$\begin{aligned} \dot{r} &= -\beta \cos\alpha, \\ r\dot{\phi} &= \beta \sin\alpha. \end{aligned}$$

*Here we ignore small correction terms from the metric since Earth's orbital radius* $r \gg \mu$ *and* $\beta$ *is a constant depending upon the affine parameter used to define the photon's path. Therefore*

$$h = r^2\dot{\phi} = \beta r \sin\alpha,$$

*and so*

$$c^2k^2 = \beta^2\cos^2\alpha + \frac{\beta^2 r^2 \sin^2\alpha}{r^2}\left(1 - \frac{2\mu}{r}\right) = \beta^2\left(1 - \frac{2\mu}{r}\sin^2\alpha\right).$$

*The photons will be captured provided that this exceeds the maximum at* $r = 3\mu$ *which is given by*

$$\frac{\beta^2 r^2 \sin^2\alpha}{(3\mu)^2}\left(1 - \frac{2\mu}{3\mu}\right).$$

*Therefore the photons are captured if*

$$1 - \frac{2\mu}{r}\sin^2\alpha > \frac{r^2\sin^2\alpha}{27\mu^2},$$

*which gives*

$$\sin\alpha < \left(\frac{27\mu^2 r}{r^3 - 54\mu^3}\right)^{1/2}.$$

*For* $r \gg \mu$ *as here, this reduces to*

$$\alpha < 3\sqrt{3}\,\frac{GM}{c^2 r},$$

*measured in radians. This works out to be* 0.011 *arcseconds. Note that* $r\sin\alpha$ *is the impact parameter, and therefore* $3\sqrt{3}\,GM/c^2$ *is the effective cross-sectional radius of a black-hole for intercepting photons. One could for example calculate the rate at which black-holes sweep up photons from the microwave background from this.*

---

**7.6**. A satellite of mass $m$ orbits an object of mass $M$ at constant distance $r$.

(a) Writing as usual $\mu = GM/c^2$, show that the energy constant $k$ is given by

$$k^2 = 1 - \left(\frac{r-4\mu}{r-3\mu}\right)\frac{\mu}{r}.$$

---

*We have*

$$c^2(k^2-1) = \dot{r}^2 + \frac{h^2}{r^2}\left(1-\frac{2\mu}{r}\right) - \frac{2\mu c^2}{r}.$$

*An orbit of constant radius implies that* $\dot{r} = 0$ *and from lectures that*

$$h^2 = \frac{\mu c^2 r^2}{r-3\mu}.$$

*(Follows from* $dV/dr = 0$*.) Therefore*

$$\begin{aligned} c^2(k^2-1) &= \frac{\mu c^2}{r-3\mu}\left(1-\frac{2\mu}{r}\right) - \frac{2\mu c^2}{r}, \\ &= \frac{\mu c^2}{r(r-3\mu)}\left(r-2\mu-2(r-3\mu)\right), \\ &= \frac{\mu c^2}{r(r-3\mu)}(-r+4\mu), \end{aligned}$$

*from which the result follows straightforwardly.*

---

(b) Hence show that at $r \gg \mu$ the energy of the particle is given by

$$E \approx mc^2 - \frac{GMm}{2r}.$$

---

*For* $r \gg \mu$

$$k^2 \approx 1 - \frac{\mu}{r},$$

*and so*

$$k \approx 1 - \frac{\mu}{2r}.$$

*Therefore*

$$E = kmc^2 = mc^2 - \frac{\mu c^2 m}{2r} = mc^2 - \frac{GMm}{2r}.$$

---

Give a physical interpretation for this result.

---

*In Newtonian terms, the final term is the sum of the potential and kinetic energies. The potential energy has twice the magnitude of the kinetic energy and so the overall result* $= -GMm/2r$*. The first term is the rest mass of the particle.*

---

(c) Obtain an expression for the ratio $\alpha$ of the rate at which time passes for the particle compared to the rate it passes for an observer stationary at infinity.

---

*The constant $k$ is defined by*

$$k = \left(1 - \frac{2\mu}{r}\right)\frac{dt}{d\tau},$$

*where $\tau$ is the proper time experienced by the orbiting particle and $t$ the coordinate time applies to the stationary observer at infinity. Thus*

$$\alpha = \frac{d\tau}{dt} = \frac{1}{k}\left(1 - \frac{2\mu}{r}\right),$$

*and therefore substituting for $k$ and applying a little algebra,*

$$\alpha = \left(\frac{r - 3\mu}{r}\right)^{1/2}.$$

---

Calculate $\alpha$ for the last stable circular orbit.

---

*In this case $r = 6\mu$, so*

$$\alpha = \frac{1}{\sqrt{2}} = 0.707,$$

*so an astronaut's clock would run at 70% of an external observer, if the astronaut was in the last stable orbit.*

*One can "time travel" in this manner more effectively if one is prepared to orbit in unstable circular orbits. These run all the way down to $r = 3\mu$ and so arbitrarily low values of $\alpha$ are possible. However, for very small $\alpha$, the cost in terms of energy to get into such an orbit would be prohibitive, and one can imagine sleepless nights with the orbit control of the spacecraft on "autopilot".*

---

Show that for $r \gg \mu$

$$\alpha \approx 1 - \frac{3GM}{2c^2 r}.$$

---

*This follows directly from the previous answer since*

$$\alpha = \left(\frac{r - 3\mu}{r}\right)^{1/2} = \left(1 - \frac{3\mu}{r}\right)^{1/2} \approx 1 - \frac{3GM}{2c^2 r},$$

*if $r \gg \mu$.*

---

Give an interpretation of this result in terms of gravitational and special-relativistic time dilation factors.

---

*We expect a* $1 + \phi/c^2 = 1 - GM/c^2 r$ *factor from gravitational time dilation and a* $(1 - v^2/c^2)^{1/2} \approx 1 - v^2/2c^2$ *factor from the usual SR time dilation. For a Newtonian circular orbit,* $v^2 = GM/r$*, and hence the* $3/2$ *factor here which can be thought of as* $2/3$ *gravitational time dilation,* $1/3$ *SR time dilation.*

---

(d) The GPS satellites provide a famous practical example of GR. The GPS satellites orbit at a radius of 26,600 km around Earth which has mass $5.98 \times 10^{24}$ kg and radius 6370 km. Calculate the rate at which the GPS satellites gain or lose compared to clocks on Earth, and hence, given that they provide positional information by timing, calculate the positional error that could result after one day of ignoring relativistic effects.

---

*A subtlety here is that we need to account for the relativistic effects on Earth as well as the satellite. The difference between the rates at which clocks run on Earth cf the satellites (scaled by an observer at infinity)* $= \alpha_E - \alpha_{GPS}$*, which, ignoring Earth's rotation, evaluates to*

$$= \frac{3 \times 6.67 \times 10^{-11} \times 5.98 \times 10^{24}}{2 \times (3.00 \times 10^8)^2 \times 2.66 \times 10^7} - \frac{6.67 \times 10^{-11} \times 5.98 \times 10^{24}}{(3.00 \times 10^8)^2 \times 6.37 \times 10^6} = 2.5 \times 10^{-10} = -4.46 \times 10^{-10}.$$

*The negative sign indicates that the Earth clocks runs slower than the satellites. In one day (86400 sec) this amounts to a drift of* 38.5 *microseconds, the equivalent of* 11 km *positional error!*

---

**7.7**. An alternative (but deeply unattractive) explanation for the 43"/century "anomalous" precession of the perihelion of Mercury is that Newton's law of gravity should be modified from $1/r^2$ to $1/r^{2+\epsilon}$ where $\epsilon$ is small but non-zero.

Calculate the value of $\epsilon$ that matches Mercury's precession rate.

---

*This is a chance to try out a slightly different version of the perturbation calculation used to derive the rate of precession in the Schwarzschild case. If Newton's law of gravity is modified as suggested then the potential will become something like*

$$\phi = -\frac{GM}{r^{1+\epsilon}},$$

*and the total energy per unit mass will be*

$$\frac{E}{m}\frac{1}{2}\left(\dot{r}^2 + (r\dot{\phi})^2\right) - \frac{GM}{r^{1+\epsilon}}.$$

*Assuming it is still a central force then* $r^2\dot{\phi} = h$ *and so the effective potential becomes*

$$V(r) = \frac{h^2}{2r^2} - \frac{GM}{r^{1+\epsilon}}.$$

*For circular orbits* $V'(r) = 0$ *so*

$$-\frac{h^2}{r^3} + (1+\epsilon)\frac{GM}{r^{2+\epsilon}} = 0.$$

*(Note there is ambiguity about whether the force or the potential should have a $1+\epsilon$ factor, but it is irrelevant for the final result.) Thus*

$$h^2 = (1+\epsilon)GMr^{1-\epsilon}. \tag{12}$$

*We will use this later to substitute for $h$, but for now we note that it implies that*

$$\dot{\phi}^2 = \frac{h^2}{r^4} = (1+\epsilon)\frac{GM}{r^{3+\epsilon}}.$$

*To derive the precession rate we considered small oscillations in $r$ which occur at*

$$\omega_r^2 = V''(r),$$

*where $V''(r)$ is the second derivative evaluated at the minimum point just calculated. This is given by*

$$V''(r) = \frac{3h^2}{r^4} - (1+\epsilon)(2+\epsilon)\frac{GM}{r^{3+\epsilon}}.$$

*Substituting for $h^2$*

$$V''(r) = \frac{3(1+\epsilon)GM}{r^{3+\epsilon}} - (1+\epsilon)(2+\epsilon)\frac{GM}{r^{3+\epsilon}} = (1-2\epsilon^2)\frac{GM}{r^{3+\epsilon}}.$$

*Therefore the precession per orbit is given by*

$$\Delta\phi = 2\pi\left(\frac{\dot{\phi}}{\omega_r} - 1\right) = 2\pi\left(\frac{1+\epsilon}{1-2\epsilon^2}\right)^{1/2} - 1) \approx \pi\epsilon,$$

*since $\epsilon$ is clearly small. For Mercury with $P = 0.24\,yr$ and an angle of 43"/century,*

$$\Delta\phi = 5.00 \times 10^{-7} \ \ rads,$$

*and so*

$$\epsilon = 1.59 \times 10^{-7},$$

*i.e. we should speak of "Newton's $1/r^{2.000000159}$ law". Luckily GR provides a more convincing explanation.*

---

**7.8**. Both special relativity and general relativity provide the opportunity to time-travel into the future. However, to do so with SR alone requires an enormous expenditure of energy in order to accelerate near to the speed of light and then to slow down.

(a) By considering the form of the effective potential of massive particles, show that it is possible to use orbital motion around a Schwarzschild black-hole to time-travel "on the cheap" with little or no expenditure of energy.

---

*The effective potential of a Schwarzschild black-hole has an inner maximum of height that depends upon the specific angular momentum. Starting from far from the black-hole with slow speed $k \approx 1$ and the "total energy" term $c^2(k^2-1) \ll c^2$. Thus if $h$ is chosen so* *that the inner maximum potential is near zero, one can get into a close orbit around the black-hole with relatively little effort. The time factor in this orbit is governed by $\dot{t}$ given by*

$$\left(1 - \frac{2\mu}{r}\right)\dot{t} = k.$$

*Since $k = 1$, $\dot{t} > 1$ and so the proper time experienced in orbit advances more slowly than the coordinate time experienced far from the black-hole.*

---

(b) Starting from a large radius, what value of $h$ in units of $\mu c$ would allow the most effective time-travel for minimal expenditure of energy.

---

*This is the critical case of Q**7.3**b with $h = 4\mu c$.*

---

(c) By what factor could one step forward into the future by these means?

---

*The maximum of the Schwarzschild potential occurs when $V'(r) = 0$, which from lectures occurs at*

$$r_C = \frac{h^2 \pm \sqrt{h^4 - 12h^2\mu^2c^2}}{2\mu c^2}.$$

*Setting $h = 4\mu c$ and taking the negative root for the inner maximum,*

$$r_C = \frac{16\mu^2c^2 - \sqrt{256\mu^4c^4 - 192\mu^2c^4}}{2\mu c^2} = 4\mu.$$

*Therefore from part (a), since $k = 1$,*

$$\dot{t} = \frac{dt}{d\tau} = \frac{k}{1 - 2\mu/r} = 2,$$

*not a huge step forward, but for free, if you have a friendly neighbourhood black-hole.*

---

**7.9.** * The equation of geodesic deviation was quoted in lectures as

$$\frac{D^2w^\alpha}{D\lambda^2} + R^\alpha{}_{\gamma\beta\delta}\,\dot{x}^\gamma\dot{x}^\delta w^\beta = 0,$$

where $\vec{W}$ is a vector separating two particles in free-fall. The first terms is the total derivative which allows for arbitrary coordinates. If we can stick to Cartesians, it reduces to $d^2w^\alpha/d\lambda^2$.

Consider two massive particles falling vertically down along a radial line towards the North pole on Earth, separated vertically by a distance $s$. Assuming non-relativistic motion, and defining the $z$-axis to be vertical show that

$$\frac{ds^2}{dt^2} = -c^2R^z{}_{tzt}\,s,$$

i.e. the two particles accelerate relative to each other by an amount proportional to their separation, providing a way to measure one of the Riemann tensor components.

---

*This question is about tidal forces, and quantifies the idea of a "small" lab in free-fall being governed by SR. Over a finite region, the effects of gravity do not disaappear but remain as tides. Here, since motion is non relativistic, only the time component of the four-velocity terms* ($\dot{x}^\gamma$, $\dot{x}^\delta$) *remains,* $= c$. *This gives the* $c^2$. *We use proper time instead of the arbitrary affine parameter* $\lambda$ *and Cartesian coordinates so that the geodesic equation can be written*

$$\frac{d^2 w^\alpha}{d\tau^2} + c^2 R^\alpha{}_{t\beta t}\, w^\beta = 0.$$

*Comparing the position of the two particles at the same time,* $w^t = 0$, *and only the* $w^z = s$ *component remains given the definition of the* $z$*-axis, so the relation further reduces to*

$$\frac{d^2 s}{d\tau^2} + c^2 R^z{}_{tzt}\, s = 0.$$

*Finally, since motion is non-relativisitic, then* $\tau \approx t$, *and the final result is obtained.*

*It has to be said that Newtonian gravity is much easier than GR when it comes to tides, but of course cannot cope with strong fields and relativisitic speeds.*

---

**7.10**. Calculate the minimum angle above the surface of a neutron star of mass $M = 2.5\,\mathrm{M}_\odot$, and radius $R = 8\,\mathrm{km}$ at which one would have to direct a laser beam in order for the light to escape the star.

[Warning: you may be tempted to write down that

$$\tan\alpha = \frac{1}{r}\frac{dr}{d\phi},$$

where $\alpha$ is the angle to the surface, but this is not quite right: think in terms of the proper distance moved given small changes in coordinates $r$ and $\phi$.]

---

*If the angle to the surface is* $\alpha$ *then* $\tan\alpha$ *is the ratio of the proper distance in the radial direction to the proper distance in the azimuthal direction. The latter is simply* $r\,d\phi$ *(for* $\theta = \pi/2$*) from the Schwarzschild metric, but the former is* $(1 - 2\mu/r)^{-1/2}\,dr$ *and so*

$$\left(1 - \frac{2\mu}{r}\right)^{-1/2} \dot{r} = \tan(\alpha) r\dot{\phi} = \tan(\alpha)\frac{h}{r}.$$

*Using this in the photon "energy equation" to substitute for* $\dot{r}$:

$$\left(1 - \frac{2\mu}{r}\right)\tan^2(\alpha)\frac{h^2}{r^2} + \frac{h^2}{r^2}\left(1 - \frac{2\mu}{r}\right) = c^2k^2,$$

*which leads to*

$$c^2k^2 = \frac{h^2}{r^2}\left(1 - \frac{2\mu}{r}\right)\frac{1}{\cos^2\alpha}.$$

*For the photon to escape, this must exceed the peak of (twice) the effective potential at* $r = 3\mu$ *which has value*

$$2V(r = 3\mu) = \frac{h^2}{27\mu^2}.$$

*Therefore we require*

$$\frac{h^2}{r^2}\left(1 - \frac{2\mu}{r}\right)\frac{1}{\cos^2\alpha} > \frac{h^2}{27\mu^2},$$

*or*

$$\cos\alpha < 3\sqrt{3}\frac{\mu}{r}\left(1 - \frac{2\mu}{r}\right)^{1/2}.$$

*For the numbers given (hypothetical since neutron stars are generally less massive than this), taking* $G\mathrm{M}_\odot/c^2 = 1.5\,\mathrm{km}$

$$\cos\alpha < 3\sqrt{3}\frac{3.0}{8.0}\left(1 - \frac{6.0}{8.0}\right) = 0.9742,$$

*so* $\alpha > 13^\circ$.

---

**7.11.** * Consider orbits of particle subject to a central attractive force of the form $F \propto r^\alpha$.

(a) Derive a condition on $\alpha$ such that a whole number $N$ of radial "epicycles" are completed within one orbit. (The orbit is then closed.)

---

*The potential is derived from* $\int F\,dr$ *so will take the form* $kr^{\alpha+1}$ *(except in the case* $\alpha = -1$*), so including the "centrifugal barrier" the effective potential will be*

$$V(r) = \frac{h^2}{2r^2} + kr^{\alpha+1}.$$

*The condition for circular orbits* $V'(r) = 0$ *gives*

$$V'(r) = -\frac{h^2}{r^3} + k(\alpha+1)r^\alpha = 0,$$

*so that*

$$h^2 = k(\alpha+1)r^{\alpha+3}.$$

*Therefore the epicyclic frequency is given by*

$$\omega_r^2 = V''(r) = \frac{3h^2}{r^4} + k\alpha(\alpha+1)r^{\alpha-1}.$$

*Substituting for* $h^2$,

$$\begin{aligned}\omega_r^2 &= 3(\alpha+1)kr^{\alpha-1} + \alpha(\alpha+1)kr^{\alpha-1},\\ &= (\alpha+1)(\alpha+3)kr^{\alpha-1}.\end{aligned}$$

*This compares with the angular frequency* $\omega_\phi = \dot{\phi} = h/r^2$ *or*

$$\omega_\phi^2 = (\alpha+1)kr^{\alpha-1}.$$

*The condition of the question is satisfied if*

$$\omega_r = N\omega_\phi,$$

*where n is a positive integer, i.e.*

$$(\alpha+1)(\alpha+3) = N^2(\alpha+1).$$

*The case $\alpha = -1$ was excluded at the start and we are left with*

$$\alpha = N^2 - 3.$$

---

(b) Comment on the values of $\alpha$ for $N = 1$ and $N = 2$.

---

*$N = 1$ gives $\alpha = -2$, i.e. the $1/r^2$ law of Newtonian gravity; $N = 2$ gives $\alpha = 1$, a linear force law leading to SHM and orbits which take the form of ellipses centred upon the centre of attraction.*

---

(c) For what values of $\alpha$ are circular orbits stable?

---

*Given the equation for $h^2$, we must have $k(\alpha+1) > 0$. Circular orbits are stable as long as $\omega_r^2 > 0$, which implies that $\alpha > -3$. The $1/r^3$ term in the Schwarzschild effective potential can be regarded as an $\alpha = -4$ force-law term; when this dominates, it leads to instability.*

---

(d) What is the precession angle per orbit for $\alpha = -1$ and is the precession in this case prograde or retrograde?

---

*In this case the potential takes the form $k \ln r$ and the effective potential is*

$$V(r) = \frac{h^2}{2r^2} + k \ln r.$$

*The condition for circular orbits $V'(r) = 0$ gives*

$$V'(r) = -\frac{h^2}{r^3} + \frac{k}{r} = 0,$$

*so*

$$h^2 = kr^2.$$

*The epicyclic frequency is thus*

$$\omega_r^2 = V''(r) = \frac{3h^2}{r^4} - \frac{k}{r^2} = \frac{2k}{r^2}.$$

*This compares to an angular frequency of*

$$\omega_\phi^2 = \frac{h^2}{r^4} = \frac{1}{2}\omega_r^2.$$

*Therefore the precession angle per orbit is*

$$2\pi\left(\frac{\omega_\phi}{\omega_r}-1\right)=2\pi\left(\frac{1}{\sqrt{2}}-1\right)=-1.84\,rads\;/\;orbit.$$

*This is retrograde precession.*

---

**7.12**. Consider the photon energy equation for $r < 2\mu$. Show that in this case $\dot{r}$ can never be zero and so photons can only travel towards smaller or larger $r$ but never switch direction.

---

*The full energy equation reads*

$$\dot{r}^2+\frac{h^2}{r^2}\left(1-\frac{2\mu}{r}\right)=c^2k^2.$$

*For $r < 2\mu$ the second term is negative. However, the right-hand side is necessarily $\geq 0$, and so we must have $\dot{r}^2 > 0$. Therefore $\dot{r} \neq 0$ and photons are stuck with whatever sign of $\dot{r}$ they start with. This is one way of seeing that photons that enter a black-hole can never escape.*

---

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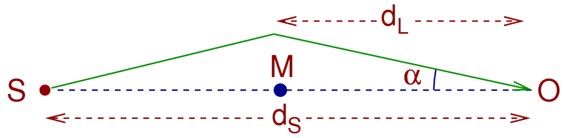
**8.1**. When a mass $M$ lies along the line of sight to a source $S$, it bends the light to form an "Einstein ring". The figure below shows the geometry of the ring formation.




(a) Use the light-deflection formula from lectures to show that the angular radius $\alpha$ is given by

$$\alpha^2 = \frac{4GM}{c^2}\left(\frac{1}{d_L} - \frac{1}{d_S}\right).$$

---

*The closest approach of the light to the mass $M$ $r_0$ is given by the small angle approximation as*

$$r_0 = d_L\alpha = (d_S - d_L)\beta,$$

*where $\beta$ is the angular radius seen from the source. Simple geometry of triangles shows that*

$$\alpha + \beta = \Delta\phi = \frac{4GM}{c^2 r_0}.$$

*Therefore substituting for $\beta$ and $r_0$,*

$$\alpha + \frac{d_L}{d_S - d_L}\alpha = \frac{4GM}{c^2 d_L \alpha},$$

*and thus*

$$\frac{d_S}{d_S - d_L}\alpha^2 = \frac{4GM}{c^2 d_L}.$$

*The relation given follows easily.*

---

(b) Calculate the angular radius of the Einstein ring formed when $M = 1\,\mathrm{M}_\odot$, $d_S = 8\,\mathrm{kpc}$ and $d_L = 4\,\mathrm{kpc}$.

---

*The angle works out to be*

$$\alpha = 4.9 \times 10^{-9}\ rad = 0.0010''.$$

*This is beyond current optical capabilities to resolve, although associated positional shifts could be measured. Easier however are the strong flux variations that occur as the source is magnified in apparent size.*

---

**8.2**. Light can orbit in a circle at radius $r = 3GM/c^2$ from a mass $M$, but such orbits are unstable. By approximating the radial '"energy" equation for photons for a small perturbation from the exact circular orbit radius, show that the perturbation grows as $e^{\phi}$ where $\phi$ is the azimuthal angle travelled by the photons.

---

*The energy equation for photon orbits is*

$$\dot{r}^2 + \frac{h^2}{r^2}\left(1 - \frac{2\mu}{r}\right) = c^2k^2.$$

*Consider the second term expanded about its maximum at $r = 3\mu$. If we define $f(r)$ by*

$$f(r) = \frac{h^2}{r^2}\left(1 - \frac{2\mu}{r}\right),$$

*then*

$$\frac{df}{dr} = -\frac{2h^2}{r^3} + \frac{6h^2\mu}{r^4} = 0,$$

*for $r = 3\mu$, while*

$$\frac{d^2f}{dr^2} = \frac{6h^2}{r^4} - \frac{24h^2\mu}{r^5} = -\frac{2h^2}{81\mu},$$

*for $r = 3\mu$. Therefore, for $r = 3\mu + \epsilon$, the energy equation can be approximated by*

$$\dot{\epsilon}^2 + \frac{1}{2}f''(r_c)\epsilon^2 = constant,$$

*since the first derivative $f'(r_c) = 0$ and ignoring third-order and higher terms. On taking the derivative, substituting for $f''$ and dividing by $2\dot{\epsilon}$*

$$\ddot{\epsilon} = \frac{h^2}{81\mu^4}\epsilon.$$

*The dots are derivatives with respect to affine parameter $\lambda$, but can be converted to something more meaningful using $h = r^2\dot{\phi}$, or $\dot{\phi} = h/9\mu^2$. If we divide this twice into both sides of the equation for $\epsilon$ then*

$$\frac{d^2\epsilon}{d\phi^2} = \epsilon.$$

*The general solution of this is $\epsilon = ae^{-\phi} + be^{\phi}$. The growing solution will soon dominate, giving growth of the type specified in the question.*

---

**8.3**. Calculate the coordinate time taken for light to complete one circular orbit of a $10\,\mathrm{M}_\odot$ black-hole.

---

*The time is given by*

$$T = \frac{2\pi}{d\phi/dt},$$

*where*

$$\frac{d\phi}{dt} = \frac{d\phi/d\lambda}{dt/d\lambda} = \frac{h(1-2\mu/r)}{r^2 k},$$

*where use has been made of* $h = r^2\dot{\phi}$ *and* $k = (1-2\mu/r)\dot{t}$. *For circular photon orbits*

$$\frac{h^2}{r^2}\left(1-\frac{2\mu}{r}\right) = c^2 k^2,$$

*therefore*

$$\frac{h}{k} = c\left(1-\frac{2\mu}{r}\right)^{-1/2} r,$$

*and so*

$$\frac{d\phi}{dt} = c\left(1-\frac{2\mu}{r}\right)^{1/2} r^{-1}.$$

*Therefore*

$$T = \frac{2\pi r}{c}\left(1-\frac{2\mu}{r}\right)^{-1/2}.$$

*For a circular orbit* $r = 3\mu$, *and remembering* $\mu = GM/c^2$, *we obtain*

$$T = 2\sqrt{3}\frac{\pi GM}{c^3}.$$

*For* $M = 10\,\mathrm{M}_\odot$, *this works out at* $5.4 \times 10^{-4}$ sec.

---

**8.4.** Were you to fall into a black-hole, would you see the singularity at $r = 0$ once you crossed the event horizon? Justify your answer.

---

*No. The singularity horizon lies in your future, while everything you see is in your past, so you would have to be clairvoyant to see the singularity.*

---

**8.5.** Obtain expressions for the distance between two points on the same radial line at radii $r_1$ and $r_2$ (Schwarzschild coordinates) from a mass $M$ in two ways:

(a) By evaluating the proper or "ruler" distance,

---

*The Schwarzschild metric is*

$$ds^2 = c^2\left(1-\frac{2\mu}{r}\right)dt^2 - \left(1-\frac{2\mu}{r}\right)^{-1} dr^2 - r^2\,d\Omega^2.$$

*Therefore the proper distance is given by*

$$\begin{aligned} d_P &= \int_{r_1}^{r_2} \frac{dr}{(1-2\mu/r)^{1/2}}, \\ &= \int_{r_1}^{r_2} \frac{r^{1/2}\,dr}{(r-2\mu)^{1/2}}. \end{aligned}$$

*Substituting* $r = 2\mu \cosh x$,

$$\begin{aligned} d_P &= 2\mu \int_{x_1}^{x_2} \cosh x \, dx, \\ &= \left(r_2^2 - (2\mu)^2\right)^{1/2} - \left(r_1^2 - (2\mu)^2\right)^{1/2}. \end{aligned}$$

---

(b) By taking $c$ times the light travel time as measured by an observer stationary at $r = r_2$.

---

*For photons* $ds = 0$, *and so for radial paths* ($d\Omega = 0$)

$$c\,dt = \pm \frac{dr}{1 - 2\mu/r}.$$

*Either sign then implies on integrating that*

$$c\Delta t = \int_{r_1}^{r_2} \frac{dr}{1 - 2\mu/r}.$$

*This is measured in coordinate time. In terms of the time of an observer at* $r_2$,

$$\Delta t_2 = \left(1 - \frac{2\mu}{r_2}\right)^{1/2} \Delta t,$$

*and therefore the light-travel time distance referred to an observer at* $r_2$ *is given by*

$$\begin{aligned} d_L &= \left(1 - \frac{2\mu}{r_2}\right)^{1/2} \int_{r_1}^{r_2} \frac{dr}{1 - 2\mu/r}, \\ &= \left(1 - \frac{2\mu}{r_2}\right)^{1/2} \int_{r_1}^{r_2} \frac{r\,dr}{r - 2\mu}, \\ &= \left(1 - \frac{2\mu}{r_2}\right)^{1/2} \int_{r_1}^{r_2} \left(\frac{r - 2\mu}{r - 2\mu} + \frac{2\mu}{r - 2\mu}\right) dr, \\ &= \left(1 - \frac{2\mu}{r_2}\right)^{1/2} \left(r_2 - r_1 + 2\mu \ln \frac{r_2 - 2\mu}{r_1 - 2\mu}\right). \end{aligned}$$

---

Hence calculate the difference between these two distances and the coordinate distance $r_2 - r_1$ for a radial line from the surface of the Sun to Earth.

[Ignore any motion of either the Sun or Earth.]

---

*In this case* $r_1 = 7 \times 10^5\,\mathrm{km}$, $r_2 = 1.5 \times 10^8\,\mathrm{km}$ *and* $\mu = 1.5\,\mathrm{km}$, *so* $r_1$ *and* $r_2 \gg \mu$. *In this case* *we can approximate the values. First, the proper distance*

$$\begin{aligned} d_P &= \left(r_2^2 - (2\mu)^2\right)^{1/2} - \left(r_1^2 - (2\mu)^2\right)^{1/2}, \\ &= r_2\left(1 - \frac{(2\mu)^2}{r_2^2}\right)^{1/2} - r_1\left(1 - \frac{(2\mu)^2}{r_1^2}\right)^{1/2}, \\ &\approx r_2\left(1 - \frac{2\mu^2}{r_2^2}\right) - r_1\left(1 - \frac{2\mu^2}{r_1^2}\right), \\ &= r_2 - r_1 + 2\mu^2\left(\frac{1}{r_1} - \frac{1}{r_2}\right). \end{aligned}$$

*This is longer than the coordinate estimate by*

$$2\mu^2\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = 4.2\,\text{mm}.$$

*For the light-travel distance, the excess over* $r_2 - r_1$ *is easily shown to be*

$$\approx 2\mu \ln\frac{r_2}{r_1} - (r_2 - r_1)\frac{\mu}{r_2}.$$

*The first term is the "Shapiro delay" term while the second simply arises from the gravitational time dilation at* $r_2$*. For the Sun–Earth path here, this works out at* 14.6 km.

---

**8.6**. In terms of Schwarzschild coordinates $r$ and $t$, Kruskal's coordinates $u$ and $v$ are given for $r > 2\mu$ by

$$\begin{aligned} v &= \left(\frac{r}{2\mu} - 1\right)^{1/2} \exp\left(\frac{r}{4\mu}\right) \sinh\left(\frac{ct}{4\mu}\right), \\ u &= \left(\frac{r}{2\mu} - 1\right)^{1/2} \exp\left(\frac{r}{4\mu}\right) \cosh\left(\frac{ct}{4\mu}\right), \end{aligned}$$

whereas for $r < 2\mu$

$$\begin{aligned} v &= \left(1 - \frac{r}{2\mu}\right)^{1/2} \exp\left(\frac{r}{4\mu}\right) \cosh\left(\frac{ct}{4\mu}\right), \\ u &= \left(1 - \frac{r}{2\mu}\right)^{1/2} \exp\left(\frac{r}{4\mu}\right) \sinh\left(\frac{ct}{4\mu}\right), \end{aligned}$$

Show that

(a) lines of constant $r$ are hyperbolae in Kruskal coordinates,

---

*The equations for* $u$ *and* $v$ *are easily combined to eliminate* $t$ *using the relation* $\cosh^2 - \sinh^2 = 1$ *which implies*

$$u^2 - v^2 = \left(\frac{r}{2\mu} - 1\right) \exp\left(\frac{r}{2\mu}\right),$$

*for* $r > 2\mu$ *and also for* $r < 2\mu$*. For constant* $r$ *these are clearly hyperbolae.*

---

(b) lines of constant $t$ are straight lines through the origin, and, in particular, that straight lines running through the origin at $\pm 45°$ represent $t = \pm\infty$, $r = 2\mu$.

---

*If one takes the ratio $u/v$, the radius $r$ drops out, so lines of constant $t$ give constant $u/v$. Lines of $45°$ require $v = \pm u$, or $\sinh(ct/4\mu) = \cosh(ct/4\mu)$. This is only possible for $t \to \pm\infty$.*

---

(c) the interval is given by

$$ds^2 = \frac{32\mu^3}{r} \exp\left(-\frac{r}{2\mu}\right)\left(dv^2 - du^2\right) - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right),$$

---

*This is best done in reverse in this case, i.e. by obtaining expressions for $du$ and $dv$ in terms of $dr$ and $dt$ and showing that they lead back to the Schwarzschild metric. We have*

$$\begin{aligned} dv \;=\; & \left(\frac{r}{2\mu} - 1\right)^{1/2} \exp\left(\frac{r}{4\mu}\right) \sinh\left(\frac{ct}{4\mu}\right) \frac{1}{4\mu}\, dr + \\ & \left(\frac{r}{2\mu} - 1\right)^{-1/2} \exp\left(\frac{r}{4\mu}\right) \sinh\left(\frac{ct}{4\mu}\right) \frac{1}{4\mu}\, dr + \\ & \left(\frac{r}{2\mu} - 1\right)^{1/2} \exp\left(\frac{r}{4\mu}\right) \cosh\left(\frac{ct}{4\mu}\right) \frac{c}{4\mu}\, dt, \end{aligned}$$

*which reduces to*

$$\begin{aligned} dv \;=\; & \frac{r}{2\mu}\left(\frac{r}{2\mu} - 1\right)^{-1/2} \exp\left(\frac{r}{4\mu}\right) \sinh\left(\frac{ct}{4\mu}\right) \frac{1}{4\mu}\, dr + \\ & \left(\frac{r}{2\mu} - 1\right)^{1/2} \exp\left(\frac{r}{4\mu}\right) \cosh\left(\frac{ct}{4\mu}\right) \frac{c}{4\mu}\, dt. \end{aligned}$$

*Similarly*

$$\begin{aligned} du \;=\; & \frac{r}{2\mu}\left(\frac{r}{2\mu} - 1\right)^{-1/2} \exp\left(\frac{r}{4\mu}\right) \cosh\left(\frac{ct}{4\mu}\right) \frac{1}{4\mu}\, dr + \\ & \left(\frac{r}{2\mu} - 1\right)^{1/2} \exp\left(\frac{r}{4\mu}\right) \sinh\left(\frac{ct}{4\mu}\right) \frac{c}{4\mu}\, dt. \end{aligned}$$

*Taking the difference between the squares $dv^2 - du^2$, the cross terms cancel and using $\cosh^2 - \sinh^2 = 1$ we are left with*

$$dv^2 - du^2 = \left(\frac{r}{2\mu} - 1\right) \exp\left(\frac{r}{2\mu}\right) \frac{c^2}{(4\mu)^2}\, dt^2 - \left(\frac{r}{2\mu}\right)^2 \left(\frac{r}{2\mu} - 1\right)^{-1} \exp\left(\frac{r}{2\mu}\right) \frac{1}{(4\mu)^2}\, dr^2.$$

*Multiplying by $16\mu^2 \exp(-r/2\mu)$*

$$16\mu^2 \exp\left(-\frac{r}{2\mu}\right)\left(dv^2 - du^2\right) = \left(\frac{r}{2\mu} - 1\right) c^2\, dt^2 - \left(\frac{r}{2\mu}\right)^2 \left(\frac{r}{2\mu} - 1\right)^{-1} dr^2.$$

*Finally multiplying by $2\mu/r$*

$$\frac{32\mu^3}{r}\exp\left(-\frac{r}{2\mu}\right)\left(dv^2 - du^2\right) = \left(1 - \frac{2\mu}{r}\right)c^2\,dt^2 - \left(1 - \frac{2\mu}{r}\right)^{-1} dr^2.$$

*The right-hand side of this equation is the first part of the Schwarzschild interval; the angular terms go through unchanged. For $r < 2\mu$, the same results.*

---

(d) $u$ is spacelike and $v$ is timelike, for all $r$

---

*"Spacelike" implies $ds^2 < 0$. For $dv = d\Omega = 0$, the form of the interval makes it clear that $u$ is spacelike. Similarly $v$ is clearly timelike (leading to $ds^2 > 0$).*

---

(e) photons on radial paths travel on the same $\pm 45°$ lines in $u$, $v$ coordinates that they do in Minkowski spacetime diagrams

---

*Photons always travel along null paths $ds^2 = 0$, and for radial paths $d\Omega = 0$, so*

$$dv^2 - du^2 = 0,$$

*which gives*

$$v = \pm u + c,$$

*QED.*

---

(f) there is one line of $r = 0$ in Kruskal space that can only ever be in your future and another that can only ever be in your past.

---

*For $r = 0$, the relation of part (a) reduces to*

$$v^2 - u^2 = 1,$$

*i.e.*

$$v = \pm\sqrt{1 + u^2}.$$

*In a $u$–$v$ spacetime diagram, with $v$ timelike (vertical axis), these are two hyperbolae as shown in Fig. 2. It can be seen that points on these hyperbolae can only ever lie in the future (upper hyperbola) or past (lower hyperbola) of events in region (1) which represent the region outside the event horizon. In fact they can only represent the past and future of any events since the regions above the upper or below the lower hyperbola do not correspond to any real events. The upper hyperbola is the singularity one cannot avoiding meeting on falling through the event horizon of a black-hole. The lower one could only ever produce particles a so-called "white hole". Whether these exist in reality is unknown.*

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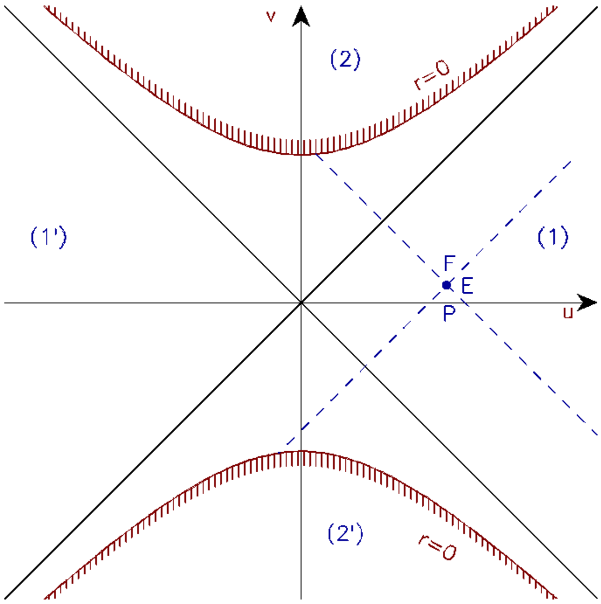


Figure 2: Kruskal diagram showing an event $E$ and its past and future light-cones..

**8.7**. During an experiment near a black-hole, Alice's jet-pack fails. Her selfish fellow astronaut Bob's main concern is for his own peace of mind in the event of his seeing Alice tested to breaking point by tidal forces.

Use a Kruskal diagram to show graphically that, however long he waits, Bob will never see Alice cross the event horizon. (Thus as long as the black-hole is massive enough to swallow Alice in one piece, Bob need not worry about future sleepless nights.)

In a fit of remorse, Bob decides that he will at least keep sending Alice signals while he can still see her, although this threatens to be forever as she appears to him to be stuck at the event horizon. Show again from the Kruskal diagram that there will come a time when Bob should stop sending signals to Alice because she will in fact have reached the singularity.

[You may assume that Alice and Bob are on the same radial line.]

---

*Fig. 3 describes the situation. Assume that Alice and Bob were at radial coordinate $r = 2.5\mu$ when Alice's jet-pack failed at event $F$, and that Bob stays at this radius so that the right-hand hyperbola is his worldline. Alice's worldline into the black-hole is marked by the dashed line, which ends at $C$. A typical signal from Alice to Bob is shown by the line $A$ to $B$. It is evident that once Alice crosses the event horizon (the $45°$ line running from the origin), no more signals from her will reach Bob. As Alice approaches the event horizon it will take longer and longer for signals from her to reach Bob who will see her frozen at the point she crosses the event horizon.*

*Alice can keep receiving signals from Bob up until event $L$ on Bob's worldline. A signal from this point will meet Alice at the same time as she reaches the singularity at $C$. There will be no point in Bob's signalling after $L$.*

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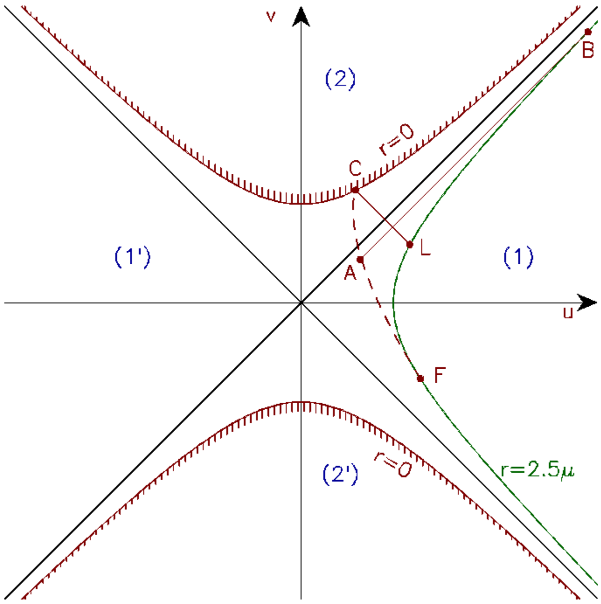


Figure 3: Kruskal diagram showing Alice's path into the black-hole, $F$ to $C$.

**8.8**. "Advanced Eddington-Finkelstein" coordinates use the constant of integration in the equation for worldlines of radially infalling photons to replace the coordinate $t$ in the Schwarzschild metric with $t'$ given by

$$ct' = ct + 2\mu \ln \left| \frac{r}{2\mu} - 1 \right| .$$

(a) Obtain an expression for the interval in the Schwarzschild geometry using $t'$ and $r$ rather than $t$ and $r$ (you may ignore the angular terms).

(b) Derive relations for $t'$ as a function of $r$ for in- and out-going photons and use these to plot the light-cone structure near a black-hole in these coordinates.

(c) Use this structure to show that once inside a black-hole, there is no escape.

---